BWeb Notes for Chapter 7: S-Shaped Growth
(two notes)

1st Note, page 80, Logistic Equation is the Solution to the Differential Equation
This note is similar to the 3rd note in Chapter 3, the note in Table 3.1 about a simpler differential equation. The differential equation for flowered area in Fig. 7.3 more complicated. Let’s borrow from Wikipedia for some background.

Wikipedia’s description of the logistic function and its value in ecology
(Modeling Population Growth)
(Wikipedia material in italics, with the links and a few other remarks removed)

A typical application of the logistic equation is a common model of population growth, (originally due to Verlunst in 1838), where the rate of reproduction is proportional to the existing population and the amount of available resources, all else being equal. Thus the second term models the competition for available resources, which tends to limit the population growth. Letting \( P \) represent population size ... and \( t \) represent time, this model is formalized by the differential equation:

\[
\frac{dP}{dt} = rP \left(1 - \frac{P}{K}\right)
\]

where the constant \( r \) defines the growth rate and \( K \) is the carrying capacity.

Interpreting the equation shown above: the early, unimpeded growth rate is modeled by the first term \( +rP \). The value of the rate \( r \) represents the proportional increase of the population \( P \) in one unit of time. Later, as the population grows, the second term, which multiplied out is \( -rP^2/K \), becomes larger than the first as some members of the population \( P \) interfere with each other by competing for some critical resource, such as food or living space. This antagonistic effect is called the bottleneck, and is modelled by the value of the parameter \( K \). The competition diminishes the combined growth rate, until the value of \( P \) ceases to grow (this is called maturity of the population).

Let us divide both sides of the equation by \( K \) to give

\[
\frac{d}{dt} \frac{P}{K} = r \frac{P}{K} \left(1 - \frac{P}{K}\right)
\]

Now setting \( x = \frac{P}{K} \) gives us the differential equation

\[
\frac{dx}{dt} = rx \left(1 - x\right)
\]

For \( r = 1 \) we have the particular case with which we started.
... The solution to the equation (with $P_0$ being the initial population) is

$$P(t) = \frac{KP_0e^{rt}}{K + P_0(e^{rt} - 1)}$$

where

$$\lim_{t \to \infty} P(t) = K.$$

Which is to say that $K$ is the limiting value of $P$: the highest value that the population can reach given infinite time (or come close to reaching in finite time). It is important to stress that the carrying capacity is asymptotically reached independently of the initial value $P(0) > 0$, also in case that $P(0) > K$.

To translate the Wikipedia analytical solution to the solution on page 80, we would replace $P(t)$ with $A(t)$ and let $A_0$ stand for the area at the start of the simulation. Then divide both the numerator and the denominator by $K$. This gives the equation on page 80:

$$A(t) = A_0 e^{rt} / \left( 1 + A_0/K \right) \left( e^{rt} - 1 \right)$$

This equation gives the same values of $A(t)$ as the simulation, provided we set $DT$ sufficiently small (as you may verify in exercise 7.7.)

The 2\textsuperscript{nd} Note is on page 81: the challenge of estimating $K$ from time series data

Imagine you are using curve fitting to estimate $K$ in a flowered system that exhibits logistic growth in an area with 1,000 total acres. The system is not identical to the model on pages 79, so we do not know that it will end up with 800 acres. Fig. 1 shows the interface of a simple model to test our ability to estimate $K$ as time series on the area becomes available. You have $800 for data collection; it costs $100 per year to gather the data; so we will bust the budget after 8 years. So, is 8 years going to be enough time? Take a look at the first 8 years of the trajectory in Fig. 7.5 and realize that you have the exact equation for the area. But we don’t have the values for the intrinsic growth rate, the decay rate and the so-called carrying capacity ($K$). Could you estimate $K$ by the 8\textsuperscript{th} year?

Fig. 1 shows a test simulation with four years of synthetic data (data from an actual model). The red line is cumulative spending; the blue line is the flowered area. At this point, we have exactly 124 acres of flowers. There is some randomness in the intrinsic growth rate (the gray curve), but the random effects do no show up strongly in the flowered area at this point. Could you estimate the value of $K$ from a visual fit at this point? (We know that the flowered area will follow a logistic trajectory, so it will eventually reach a limit, and the exercise hints that the $K$ will correspond to one of the 6 information buttons arranged on the right side of the graph. However, at this point, I suspect few readers would venture a guess as to which of the six buttons is the best representation of $K$.)
Fig. 1. Flowered area with four years of simulation results.

Fig. 2 shows the situation two years later. We have six years of data and we have expended $600 of the $800 budget. The random effects have been more dramatic in the past two years, so the flowered area is no longer on a smooth trajectory. The area is now 353 acres, and we have exact knowledge of the intrinsic growth rates. Could you estimate $K$ at this point in the test?

Fig. 2. Flowered area with six years of simulation results.
Fig. 3 concludes the simulation experiment so you see that flowered area will eventually reach an approximate equilibrium value similar to values shown in the book. By the 14\textsuperscript{th} year, the area is hovering around 880 to 890 acres, and the 2\textsuperscript{nd} of the six buttons is the best visual indicator of $K$.\footnote{If this test were repeated with the toggle switch on eliminate randomness turned on, we would see the familiar logistic curve, and it would reach equilibrium of 882 acres by around the 12\textsuperscript{th} year.}

This exercise shows the difficulty in “visual curve fitting” to estimate $K$ in advance. By the time we have enough data to tell us that $K$ is around 880 acres, the system has already reached 880 acres!

Testing Statistical Curve Fitting with Synthetic Data

But perhaps statistical methods would allow one to estimate $K$ much more accurately and much sooner. You can test this proposition with the model used in this exercise. Simply run the model with your own assumptions on the parameters.\footnote{The parameters in this test were: total area = 1,000 acres; starting area = 10 acres; and the multiplier = the linear shape shown in Fig. 7.4. So far, everything is the same as the book. But this test changes the decay rate from 0.20/yr to 0/10/yr, and it changes the intrinsic growth rate from 1.0/yr to 0.85/yr. These changes have the effect of changing the “r” in the logistic equation from 0.80/yr to 0.75/yr.} The results for flowered area may be called “synthetic data” since they represents data from a system that we have constructed. Ask Stella for a table of results for area; export the table values to a spreadsheet; and apply your favorite curve fitting method to estimate the value of $K$. 

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