1. (a) $L$ is lower triangular with 1’s on the diagonal. $U$ is upper diagonal with pivots on the diagonal.

(b) No. If a zero pivot is encountered during Gaussian elimination, then a nonsingular matrix will not have a $LU$ factorization. (However, a permuted version of the matrix would have a $LU$ factorization.)

(c) If the $LU$ factorization of a matrix is known, solving the linear system $Ax = b$ using this factorization involves far fewer arithmetic operations than Gaussian elimination.

(d) $Ax = (LU)x = L(Ux) = Ly = b$, where $Ux = y$. To solve $Ax = b$, first solve $Ly = b$ (forward substitution): $y_1 = b_1 = 1; 2y_1 + y_2 = 2(1) + y_2 = b_2 = -1 \Rightarrow y_2 = -3$.

Next, solve $Ux = y$ (back substitution): $3x_2 = y_2 = -3 \Rightarrow x_2 = -1. 2x_1 - x_2 = 2x_1 - (-1) = y_1 = 1 \Rightarrow 2x_1 = 0 \Rightarrow x_1 = 0$. \hspace{1cm} \therefore \hspace{0.5cm} x = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.

2. (a) Multiply both sides of $A(I - A) = (I - A)A$ on the left by $(I - A)^{-1} \Rightarrow$

\[(I - A)^{-1}A(I - A) = A. \] Now multiply both sides by $(I - A)^{-1}$ on the right \Rightarrow

\[(I - A)^{-1}A = A(I - A)^{-1}.\]

(b) $B^T = [(I + K)(I - K)^{-1}]^T = [(I - K)^{-1}]^T(I + K) = [(I - K)^T]^{-1}(I + K)^T$

\[= (I^T - K^T)^{-1}(I^T + K^T) = (I + K)^{-1}(I - K)\]

$B^{-1} = [(I + K)(I - K)^{-1}]^{-1} = [(I - K)^{-1}]^{-1}(I + K)^{-1}$

\[= (I - K)(I + K) \overset{(a)}{=} (I + K)(I - K) = B^T\]
3. (a) Use Gaussian elimination to solve $Ax = 0$:

$$
\begin{pmatrix}
1 & 1 & -1 & 0 \\
0 & 1 & 1 & 0 \\
2 & 3 & -1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & -1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & -1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

Now solve for the basic variables ($x_1$ and $x_2$) in terms of the free variable $x_3$:

$$x_2 = -x_3, \quad x_1 + x_2 - x_3 = x_1 + (-x_3) - x_3 = x_1 - 2x_3 = 0 \Rightarrow x_1 = 2x_3.$$ 

$$x = \begin{pmatrix} 2x_3 \\ -x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \text{a basis for the nullspace of } A \text{ is } \{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \}.$$

(b) $\dim R(A^T) = \text{rank}(A^T) = \text{rank}(A) = 2$ since the row echelon form of $A$ has two pivots (see underlined elements above).

4. $\tilde{S} = \{ w, z \}$ is linearly independent $\Leftrightarrow \alpha_1 w + \alpha_2 z = 0 \Rightarrow \alpha_1 = \alpha_2 = 0$. Now

$$\alpha_1 w + \alpha_2 z = \alpha_1 (x + y) + \alpha_2 (x - y) = (\alpha_1 + \alpha_2) x + (\alpha_1 - \alpha_2) y = 0.$$ 

The coefficients of $x$ and $y$ must both vanish since $S$ is a linearly independent set. That is, $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 - \alpha_2 = 0 \Rightarrow \alpha_1 = -\alpha_2$ and $\alpha_1 = \alpha_2 \Rightarrow \alpha_2 = -\alpha_2 \Rightarrow \alpha_2 = 0 \Rightarrow \alpha_1 = 0$.

Thus, $\alpha_1 w + \alpha_2 z = 0 \Rightarrow \alpha_1 = \alpha_2 = 0$ and therefore, $\tilde{S}$ is a linearly independent set.

5. Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix}$. Then $v \in \text{span}(S) \Leftrightarrow Ax = v$ is consistent $\Leftrightarrow \text{rank}(A|v) = \text{rank}(A)$.

Now $(A|v) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & c \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & c+3 \end{pmatrix}$. Thus, $\text{rank}(A|v) = \text{rank}(A) \Leftrightarrow c + 3 = 0 \Leftrightarrow c = -3$. Thus, $v \in \text{span}(S)$ if and only if $c = -3$. 

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