Key to Problem Set #4

1. (a) Obviously, $|f(t)| \geq 0$ for all $t \in [0, 1] \Rightarrow |f| \geq 0$. If $f(t) = 0$ for all $t \in [0, 1]$ then clearly $|f| = 0$. If $|f| = 0$, then $0 = \max_{t \in [0,1]} |f(t)| \geq |f(t)| \geq 0 \Rightarrow f(t) = 0$ for all $t \in [0,1] \Rightarrow f(t) = 0$. If $f(t) = 0$, then $0 = \max_{t \in [0,1]} |f(t)| \geq |f(t)| \geq 0 \Rightarrow f(t) = 0$ for all $t \in [0,1]$. Thus, $f(t) = 0 \iff f(t) = 0$. Now let $\alpha$ be a real scalar and consider $\alpha f = \max_{t \in [0,1]} |\alpha f(t)| = \alpha \max_{t \in [0,1]} |f(t)| = |\alpha| |f|$. Finally, $\alpha f + \alpha g = \max_{t \in [0,1]} |f(t)| + |g(t)| \leq \max_{t \in [0,1]} |f(t)| + |g(t)| = f + g$ which proves the triangle inequality.

(b) There is an inner product if and only if $\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2)$ for all $f$ and $g$ in $V$. Consider $f(t) = 1$ and $g(t) = t$. $\|f + g\|^2 = 4$, $\|f - g\|^2 = 1$, $\|f\| = 1$, and $\|g\| = 1$. Thus $\|f + g\|^2 + \|f - g\|^2 = 5 \neq 2(\|f\|^2 + \|g\|^2) = 4$ which implies that there is no inner product on $V$ such that $\langle f, f \rangle = \|f\|^2$ for all $f \in V$.

2. (a) $\theta = \arccos \left( \frac{\int_0^1 (1+t)(1+t^2) dt}{\sqrt{\int_0^1 (1+t)^2 dt} \cdot \sqrt{\int_0^1 (1+t^2)^2 dt}} \right) = \arccos \left( \frac{25/12}{\sqrt{7/3} \cdot \sqrt{28/15}} \right) \approx .059$ radians.

(b) $v_1(t) = 1$, $v_2(t) = \sqrt{3}(2t - 1)$, and $v_3(t) = \sqrt{5}(6t^2 - 6t + 1)$.

(c) $1 + t^2 = \frac{4}{3} + 1 + \frac{1}{2\sqrt{3}} \cdot \sqrt{3}(2t - 1) + \frac{1}{6\sqrt{5}} \cdot \sqrt{5}(6t^2 - 6t + 1)$.

3. (a) Let $v \in V_0$ and let $\alpha$ be a nonzero scalar. Then $\langle \alpha v, p \rangle = \alpha \langle v, p \rangle = 0 \Rightarrow \alpha v \in V_0$. Suppose $v, w \in V_0$. Then $\langle v + w, p \rangle = \langle v, p \rangle + \langle w, p \rangle = 0 \Rightarrow v + w \in V_0$.

Therefore, $V_0$ is a subspace since it is closed under scalar multiplication and vector addition.

(b) Let $B = \{u_1, u_2, \ldots, u_n\}$ be a basis for $V$ and assume that $\langle p, u_1 \rangle \neq 0$. (This must be true for some basis vector since $p \in V$, and we are thus assuming without loss of generality that $B$ is ordered so that $p$ is not orthogonal to the first basis vector.)
Since $B$ is a basis for $V$ and $\{p, u_1\} \neq 0$, we can use the Gram-Schmidt process starting with the set $B$—but with $p$ replacing $u_1$—to produce an orthonormal basis $B'$ for $V$: $B' = \left\{ \frac{p}{\|p\|}, v_1, v_2, \ldots, v_{n-1} \right\}$, where the $v_i$ are the vectors generated by the Gram-Schmidt procedure. Claim: the set $B_0 = \{v_1, v_2, \ldots, v_{n-1}\}$ is a basis for $V_0$.

This immediately implies that $\dim(V_0) = n - 1$. To prove the claim, we need to show that $B_0$ is linearly independent and spans $V_0$. The linear independence follows from the fact that $B_0$ is, by construction, an orthonormal set. To show that $B_0$ spans $V_0$, consider $w \in V_0$. Since $w$ is also in $V$ and $B'$ is an orthonormal basis for $V$, the Fourier expansion of $w$ with respect to $B'$ is

$$w = \langle w, \frac{p}{\|p\|} \rangle \frac{p}{\|p\|} + \langle w, v_1 \rangle v_1 + \langle w, v_2 \rangle v_2 + \cdots + \langle w, v_{n-1} \rangle v_{n-1}$$

because $\langle w, p \rangle / \|p\|^2 = 1 / \|p\|^2 \langle w, p \rangle = 0$. Thus, every $w \in V_0$ can be written as a combination of vectors in $B_0 \Rightarrow V_0 = \text{span}(B_0)$. 