Spatial (and other) models in mathematical biology

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Well-mixed populations: ODE’s

Good model if pop sizes are large and everything is well mixed (e.g., chemostat). No spatial structure and randomness averages out.

- Single-species density: \( u(t) \)

\[
\frac{du}{dt} = ru \left(1 - \frac{u}{K}\right) \quad \text{(logistic growth)}
\]

- \( r = \text{intrinsic growth rate; } K = \text{carrying capacity} \)

- \( u(t) \rightarrow K, \text{ as } t \rightarrow \infty \)
• Multi-species densities: $u_i(t), i = 1, 2, \ldots, n$

$$\frac{du_i}{dt} = u_i \left( r_i + \sum_j a_{ij} u_j \right) \quad \text{(Lotka-Volterra models)}$$

\begin{array}{ccc}
 a_{ij} & a_{ji} \\
 - & - & \text{competitive} \\
 + & - & \text{predator \& prey} \\
 + & + & \text{mutualistic} \\
\end{array}
Spatial dependence / local mixing: PDE’s

- Intra- and inter-species interactions (as before)
- Fast *local movement*, but not global mixing
  (Ex: random motion of cells; diffusion of individuals in population)

Get some spatial structure (smoothed out and nonrandom)
Single species

\[ u(x, t) = \text{density at position } x \text{ at time } t \]

\[ \frac{\partial u}{\partial t} = \Delta u + ru\left(1 - \frac{u}{K}\right) \quad \text{(diffusion + logistic growth)} \]

“Fisher’s equation”

• spatial spread of advantageous allele or epidemic
• traveling wave front
Spatial biofilm structure; *P. putida* (red), *Acinetobacter* (purple), with transconjugants (green and yellow)
Interacting Particle Systems = CA models

- Explicitly model
  1. discrete (not smoothed out) spatial structure
  2. randomness
  3. local interactions
- Stochastic spatial simulator WinSSS (Grant Guan)
Basic set-up

• Sites on grid or “checkerboard”

• Each site can be in several different states

• Specify local interactions: At what rate does a site in state i change to state j (based on what’s in neighborhood)?
Picture of grid and neighborhoods
Ex. 1. Contact process

2 states: vacant = 0, occupied = 1
Ex 2: Epidemic model

3 states: Susceptible, Infective, Removed (dead)

Non-spatial (mass-action) model

\[
\frac{dS}{dt} = -\beta SI + \cdots
\]

\[
\frac{dI}{dt} = \beta SI - \delta I + \cdots
\]

* Spatial simulations *
some applications to microbiology

bacterial plasmids

viruses (phage)
Plasmids (with Eva Top)

- Horizontal gene transfer in bacterial communities (antibiotic resistance)
- Extrachromosomal DNA can transfer quickly between members of same species and different species
- Rapid response to environmental selective pressure
Conjugation mechanisms

1. Self-transfer of Tra\(^+\) plasmids

**Diagram:**
- **Donor**
  - Tra\(^+\) plasmid
  - Chromosome
- **Recipient**
- **Transconjugant**
differential equations for liquid culture

\[
\frac{dR}{dt} = \psi_R R + \psi_T \tau T - \gamma_T RT - u_R R \quad \text{(recipients)}
\]

\[
\frac{dT}{dt} = \psi_T (1 - \tau) T + \gamma_T RT - u_T T \quad \text{(transconjugants)}
\]

- \(\psi\) . . . growth rates (possibly depending on current density and nutrient concentration)
- \(\gamma\) . . . plasmid transfer (conjugation) rate
- \(\tau\) . . . segregation probability
- \(u\) . . . death or washout rates
effects of spatial structure

• most bacteria live attached to surfaces (e.g., biofilms)

• contact is essential for plasmid transfer (conjugation)

• Is transfer of antibiotic resistance genes different from what is predicted by mass-action differential equations?

• Should antibiotic dosing regimens take this into account to slow down the spread of resistance (and the loss of effective antibiotics)?
Spatial patterns—experiment and simulation

A. *E. coli* K12(pB10::rfp) 30°C
- White sectors
- RFP
- Segregation rate: 0.005
- Growth rate ratio: 0.95

B. *E. coli* K12(pB10::rfp) 37°C
- White sectors
- RFP
- Segregation rate: 0.005
- Growth rate ratio: 0.65

C. *Ochrobactrum sp.* LDG6(pB10::rfp) 30°C
- White sectors
- RFP
- Segregation rate: 0.5
- Growth rate ratio: 0.9 or 0.65

D. *P. putida* H2(pB10::rfp) 30°C
- White sectors
- RFP
- Segregation rate: 0.0005
- Growth rate ratio: 0.65
Phage

- (Bacterio)phages are viruses that infect bacterial cells
- Great experimental system for studying evolution of viruses
- Effect of spatial structure
- (Opening for undergraduate researcher in my phage lab this semester–krone@uidaho.edu)

* Spatial simulations *