Fall 2006
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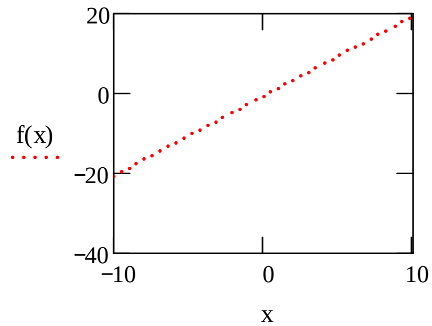
Mathematical Review Problems

I. Polynomial Equations and Graphs (Barrante--Chap. 2)

1. First degree equation and graph

$y = f(x) = m \cdot x + b$ where m is the slope of the line and b is the line's intercept

example 1 $f(x) := 2 \cdot x - 1$ x range is -10 to 10



slope m = 2

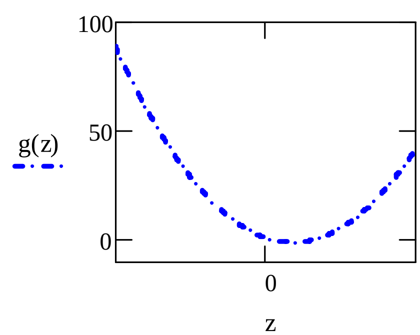
intercept b = -1

2. Second degree equation and graph

$y = f(x) = a \cdot x^2 + b \cdot x + c$

$$root1 = \frac{-b + \sqrt{b^2 - 4a \cdot c}}{2 \cdot a} \qquad root2 = \frac{-b - \sqrt{b^2 - 4a \cdot c}}{2 \cdot a}$$

example 2 $g(z) := z^2 - 3 \cdot z + 1$



1. Find the roots by using the quadratic formula

$$z1 := \frac{-(-3) + \sqrt{(-3)^2 - 4 \times 1 \cdot 1}}{2 \cdot 1} \qquad z1 = 2.618$$

$$z2 := \frac{-(-3) - \sqrt{(-3)^2 - 4 \times 1 \cdot 1}}{2 \cdot 1} \qquad z2 = 0.382$$

2. By using MathCAD biult in "**root**" function

$z := 0$ initial guess value relatively close to the roots

$$\text{root}\left[\left(z^2 - 3 \cdot z + 1\right), z\right] = 0.382$$

3. Using the MathCAD "**Given**" and " **Find**" solving block

$Z := 6$

Giver

$$Z^2 - 3 \cdot Z + 1 = 0$$

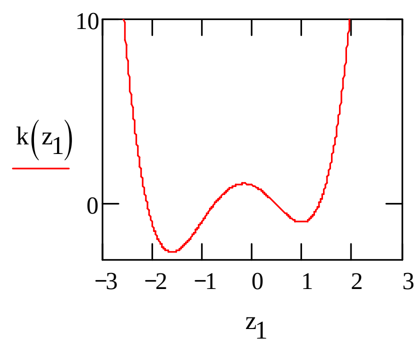
$\text{Find}(Z) = 2.618$

2. Roots of Polynomial equations

$y = h(v) = a \cdot v^4 + b \cdot v^3 + c \cdot v^2 + d \cdot v + e$ Finding the roots of h(v) could be very difficult

1. The standard way is to gragh the function

exampl€ $k(z_1) := z_1^4 + z_1^3 - 3 \cdot z_1^2 - z_1 + 1$



The roots are the value of v for which k(v) is zero

2. Use MathCAD "Symbolics" keyword **coeffs** from the **Symbolic** toolbar and the built in"polyroots" function

$$Q := k(z_1) \text{ coeffs}, z_1 \rightarrow \begin{pmatrix} 1 \\ -1 \\ -3 \\ 1 \\ 1 \end{pmatrix}$$

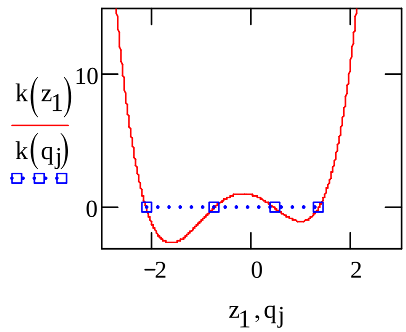
value of e

value of d

$q := \text{polyroots}(Q)$

$$q^T = \begin{pmatrix} -2.095 & -0.738 & 0.477 & 1.356 \end{pmatrix} \qquad q = \begin{pmatrix} -2.095 \\ -0.738 \\ 0.477 \\ 1.356 \end{pmatrix}$$

$j := 0, 1 .. 3$



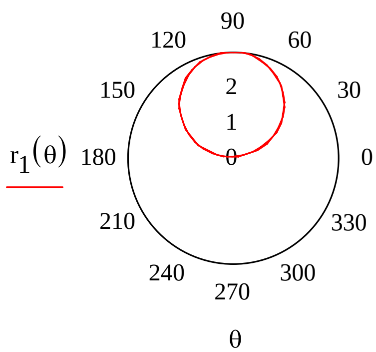
Plot the following in plane polar corrdinates from 0 to 2π.

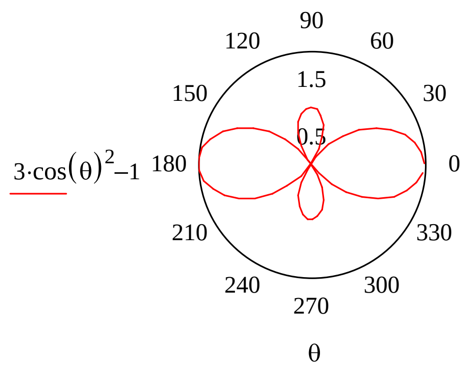
Examples

$\theta := 0, .1 .. 6.28$

$r_1(\theta) := 3 \cdot \tan(\theta) \cdot \cos(\theta)$

$r_2(\theta) := 3 \cdot \cos(\theta)^2 - 1$





Problem 1 (Barrante)

1.(b,i): Determine the roots of the equations and state whether in each case the roots are the zeros of the function.

$2(y - 2) = -6x$ (Go to "**Symbolic**" toolbar and select "**solve**")

$-x^2 + y = 5 \cdot x - 6$

3.(b,e): Plot the following functions in Cartesian coordinates

$y = 3x - 9$

$PV = k$ k is a constant

Find the roots by the graph method and then by calculation

$y1 = x^2 - 6x + 3$ Find the roots usinf the graph and the "**root"** function

Problem 2
(2-1 (i,j), Barrante) Determine the roots of the following equations, and then state whether in each case the roots are the zeros of the function

2-1(i) $x^2 + y^2 = 4$

2-1(j) $(x - 2)^2 + (y + 4)^2 = 9$

2-2 (b,e), Barrante) Plot the following functions in plane polar coordinates from 0 to 2 π

2-2 (b) $r = 4 \cdot \sin\theta$

2-2 (e) $r = 4 \cdot \sin\theta\cos\theta$

II. Partial Derivatives (Barrante--Chap. 4)

Problem 3 -- 4-1 (n.q)

4.1(n, q)
Differentiate the following functions. (All upper case letters are constants.)

(n) $e = \frac{E^2}{A} \cdot \left(z^2 - \frac{27}{8} \cdot z \right)$ <-- Use **<cntrl>=** in equation. Set cursor at z, pull down S**ymbolics** menu, select differentiate on variable.

Then solve for z using again the **Symbolics** menu; select variable and solve with the editing cursor on z

$f(z) = \frac{E^2}{A} \cdot \left(z^2 - \frac{27}{8} \cdot z \right)$ **<--** Repeat using new function that explicitly depends on z.

(q) $d = 3 \cdot \cos(\theta)^2 - 1$

Then solve for θ using again the **Symbolics** menu; select variable and solve with the editing cursor on θ

III. Integral Calculus (Barrante--Chap. 5)

Two major approaches to integral calculus.

1. Consider the integral as an antiderivative (inverse of differentiation).

$y = f(x)$ therefore $dy/dx = df(x)/dx = f(x)' = y'$

$dy = f(x)'\,dx$ *(differential form)*

The question to pose is what function f(x) when differentiated gives f(x)' the first derivative?

The function f(x) is called the <u>integral</u> *of the differential and is noted*

$$f(x) = \int f(x)'\,dx$$

2. Consider the integral as the sum of many similar, infinitesimal elements (area under the curve). This assigns a physical meaning to the integral.

For a more complete treatement of Integral Calculus you are encouraged to read Barrante Chapter 5

Problem 4 -- 5-1(a), 5-2 (a) and 5-3 (h)

5-1 (a) $\int 5 \cdot x^3\,dx$

5-2 (a) $\int e^{-4x}\,dx$

5-3 (h) for a equals 2, 5

$$\int_0^{\infty} x^2 \cdot e^{-(a \cdot x)^2}\,dx$$

IV. Matrices and Determinants (Barrante--Chap 9)

A matrix is defined as a 2-D array of numbers. It can contained an equal numbers of rows and columns (rectangle) or different number of rows and columns and it is called an *mxn matrix.* The simplest matirx is a column or a row matrix with one column or one row.

example

$a := 1$ $b := 2$ $c := 3$ $d := 4$

$$M := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

deteminant is obtained by *a x d - b x c* where a, b, c and d are the elements of the matrix determinant : $a d - b \cdot c = -2$ **MathCAD built in function** $|M| = -2$

Matrix Algebra

Interchanging rows and columns does not change the deteminant of the matrix.
Interchanging any 2 rows or columns will change the sign of the determinant.
The determinant of a square matrix is zero if any 2 rows or 2 columns are identical.
The determinant is multiplyed by k if any row or column is multiplied by a number k.
Two matrices are added by adding their elements.
Two matrices are multiplied as shown below

Exampl $M_1 = \begin{pmatrix} a1 & a2 \\ b1 & b2 \end{pmatrix}$ $M_2 = \begin{pmatrix} c1 & c2 \\ d1 & d2 \end{pmatrix}$

Additio $M_1 + M_2 = \begin{pmatrix} a1 + c1 & a2 + c2 \\ b1 + d1 & b2 + d2 \end{pmatrix}$

Multiplicatio $M_1 \cdot M_2 = \begin{pmatrix} a1 \cdot c1 + a2 \cdot d1 & a1 \cdot c2 + a2 \cdot d2 \\ b1 \cdot c1 + b2 \cdot d1 & b1 \cdot c2 + b2 \cdot d2 \end{pmatrix}$

Solving linear equations

Exampl

$x + y + z = 2$

$3 \cdot x + y - 2 \cdot z = -5$

$2 \cdot x - y - 3 \cdot z = -5$

This set of equation can also be represented by the product of 2 matrices

$M_3 := \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & -1 & -3 \end{pmatrix}$ coefficient of x, y and z $M_4 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ product $M_3 \cdot M_4 = \begin{pmatrix} 2 \\ -5 \\ -5 \end{pmatrix}$

a) calculate the determinant of the coefficient $D := |M_3|$

$D = -5$

b) calculate the determinant of the M_3 after substituing the first column with 2, -5, -5

$M'_3 := \begin{pmatrix} 2 & 1 & 1 \\ -5 & 1 & -2 \\ -5 & -1 & -3 \end{pmatrix}$ $D1 := |M'_3|$ $D1 = -5$

c) calculate the determinant of the M_3 after substituing the second column with 2, -5, -5

$$M''_3 := \begin{pmatrix} 1 & 2 & 1 \\ 3 & -5 & -2 \\ 2 & -5 & -3 \end{pmatrix} \qquad D2 := |M''_3| \qquad D2 = 10$$

d) calculate the determinant of the M_3 after substituing the third column with 2, -5, -5

$$M'''_3 := \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & -5 \\ 2 & -1 & -5 \end{pmatrix} \qquad D3 := |M'''_3| \qquad D3 = -15$$

Finally $\quad x := \frac{D1}{D} \quad x = 1 \qquad y := \frac{D2}{D} \quad y = -2 \qquad z := \frac{D3}{D} \quad z = 3$

Problem 6

Matrix algebra. Given the following matrices:

$$A := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 16 \\ 3 & 9 & 27 & 81 \\ 4 & 16 & 64 & 256 \end{pmatrix}, \qquad B(x) := \begin{pmatrix} x & 1 & 0 & -\frac{1}{2} \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ -\frac{1}{2} & 0 & 1 & x \end{pmatrix} \qquad z = \begin{pmatrix} r \\ s \\ t \\ u \end{pmatrix}$$

(a) Compute the determinant $|A|$.

(b) Find the roots of the polynomial equation $|B(x)| = 0$.

(c) Solve for the components of z when $A \cdot z = \begin{pmatrix} -4 \\ -2 \\ 0 \\ 2 \end{pmatrix}$.

V. Differential Equations (Barrante--Chap 7)

Format required to solve a differential equation or a system of differential equations using one of the command-line differential equation solvers such as

rkfixed, *Rkadapt*, *Radau*, *Stiffb*, *Stiffr* or *Bulstoer*.

For a numerical routine to solve a differential equation (DE), we must somehow pass the differential equation as an argument to the solver routine. A standard form for all DEs will allow us to do this.

Ordinary differential equations (first order).. to solve this type of DE the solving block shown below should be used (*Given and Odesolve*)

Given

y(x) = f(x)   initial condition y(x0) = 1 for instance

y:= Odesolve (x, x_{end})

plot the solution

The idea is to change the n-th order ODE into a system of n coupled first-order differential equations.

Basic idea: get rid of any second, third, fourth, etc. derivatives that appear, leaving only first derivatives.

Example:

$$\frac{d^3}{dx^3}y - \left(\frac{d}{dx}y\right)^2 \cdot \left(x^2 - 1\right) - 2 \cdot x \cdot y - e^{-500 \cdot x^2} = 0$$

This is third order, has non constant coefficients and is neither linear nor homogeneous. To solve it we
will introduce 3 new functions:

$$A(x) = y, \quad B(x) = y', \quad C(x) = y''.$$

Then the derivatives of A, B, C are simply given by

$$\frac{d}{dx}A = B, \qquad \frac{d}{dx}B = C.$$

And the derivative of C can be found by solving the given differential equation:

$$\frac{d}{dx}C = 2 \cdot x \cdot A + \left(x^2 - 1\right) \cdot B^2 + e^{-500 \cdot x^2}$$

Suppose the initial conditions are: y(0)=1, y'(0)=0, y''(0)=-1. We will find and plot the values of y(x) in the interval 0<x<4.

$$y := \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad D(x,y) := \begin{bmatrix} y_1 \\ y_2 \\ 2 \cdot x \cdot y_0 + \left(x^2 - 1\right) \cdot \left(y_1\right)^2 + e^{-500 \cdot x^2} \end{bmatrix}$$

$r := 2000$ *2000 steps*

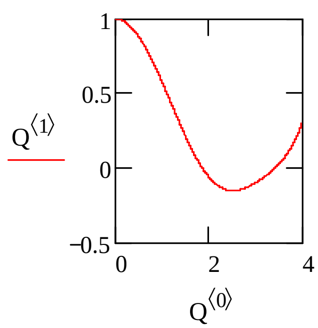
$x0 := 0$ $x1 := 4$ *the interval*

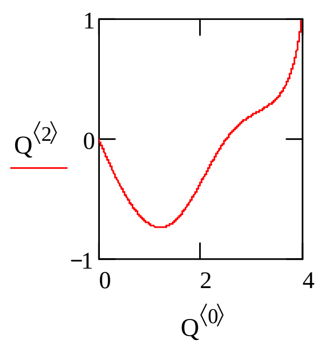
The integration is defined as follows: 5 arguments, **y** which is a vector (here a 3x1 matrix) with the BC of the solution and the first and second derivative at the initial value, then **x0** and **x1** which represent the initial and final values of **x**, **r** the number of points between initial and final values and finally the vector **D** that contains the first and the second derivative of the solution

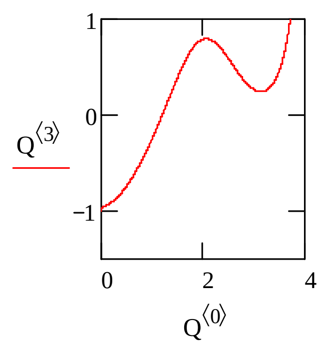
$i := 0..r$ *index for steps*

$Q := \text{rkfixed}(y, x0, x1, r, D)$ *Runge-Kutta integration*

Plot the solution its derivative its second derivative







x y y' y''

Q =

	0	1	2	3
0	0	1	0	-1
1	$2\cdot10^{-3}$	1	$-1.998\cdot10^{-3}$	-0.998
2	$4\cdot10^{-3}$	1	$-3.992\cdot10^{-3}$	-0.996
3	$6\cdot10^{-3}$	1	$-5.982\cdot10^{-3}$	-0.994
4	$8\cdot10^{-3}$	1	$-7.968\cdot10^{-3}$	-0.992
5	0.01	1	$-9.95\cdot10^{-3}$	-0.99
6	0.012	1	-0.012	-0.988
7	0.014	1	-0.014	-0.986
8	0.016	1	-0.016	-0.984

Problem 5 (7-1 (a, c, e))

(a) $\frac{d}{dx}y + 3\cdot y = 0$ interval 0< u < 10 nstep := 100

initial conditions y(0) = 1

(c) $\frac{d^2}{dx^2}y + 2\left(\frac{d}{dx}y\right) + y = 0$ interval 0 < u < 10 r := 1000 number of steps

initial conditions y(0) = 1, y'(0) = 1 and y'' (0) = 0

(e) $\frac{d^2}{dx^2}y - 9\cdot y = 0$ interval 0 < u< 5 n := 500

initial conditions y(0) = -1, y'(0) = 0 and y''(0) = 1