I. Polynomial Equations and Graphs (Barrante--Chap. 2)

1. First degree equation and graph

\[ y = f(x) = mx + b \]

where \( m \) is the slope of the line and \( b \) is the line's intercept

Example 1: \( f(x) := 2x - 1 \)

- \( x \) range is -10 to 10
- slope \( m = 2 \)
- intercept \( b = -1 \)

2. Second degree equation and graph

\[ y = f(x) = ax^2 + bx + c \]

\[ \text{root}_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

\[ \text{root}_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

Example 2: \( g(z) := z^2 - 3z + 1 \)

1. Find the roots by using the quadratic formula

\[ z_1 := \frac{-(-3) + \sqrt{(-3)^2 - 4 \times 1 \times 1}}{2 \cdot 1} \]

\[ z_1 = 2.618 \]
\[ z_2 := \frac{-(-3) - \sqrt{(-3)^2 - 4 \times 1 \times 1}}{2 \times 1} \quad z_2 = 0.382 \]

2. By using MathCAD built in "root" function
\[ z := 0 \quad \text{initial guess value relatively close to the roots} \]
\[ \text{root} \left( \left( z^2 - 3 \cdot z + 1 \right), z \right) = 0.382 \]

3. Using the MathCAD "Given" and "Find" solving block
\[ Z := 6 \]
\[ \text{Given} \]
\[ Z^2 - 3 \cdot Z + 1 = 0 \]
\[ \text{Find}(Z) = 2.618 \]

2. Roots of Polynomial equations

\[ y = h(v) = a \cdot v^4 + b \cdot v^3 + c \cdot v^2 + d \cdot v + e \quad \text{Finding the roots of } h(v) \text{ could be very difficult} \]

1. The standard way is to graph the function

\[ k(z_1) := z_1^4 + z_1^3 - 3 \cdot z_1^2 - z_1 + 1 \]

The roots are the value of \( v \) for which \( k(v) \) is zero

2. Use MathCAD "Symbolics" keyword \texttt{coeffs} from the Symbolic toolbar and the built in"polyroots" function

\[ Q := k(z_1) \text{ coeffs}, z_1 \rightarrow \begin{pmatrix} 1 \\ -1 \\ -3 \\ 1 \\ 1 \end{pmatrix} \quad \text{value of } e \]
\[ \text{value of } d \]

\[ q := \text{polyroots}(Q) \]
\[ q^T = \begin{pmatrix} -2.095 \\ -0.738 \\ 0.477 \\ 1.356 \end{pmatrix} \quad q = \begin{pmatrix} -2.095 \\ -0.738 \\ 0.477 \\ 1.356 \end{pmatrix} \]
Plot the following in plane polar coordinates from 0 to 2π.

Example:

\[ \theta := 0, 1..6.28 \]

\[ r_1(\theta) := 3 \cdot \tan(\theta) \cdot \cos(\theta) \]

\[ r_2(\theta) := 3 \cdot \cos^2(\theta) - 1 \]

**Problem 1 (Barrante)**

1.(b,i): Determine the roots of the equations and state whether in each case the roots are the zeros of the function.

\[ 2(y - 2) = -6x \]  
(Go to "Symbolic" toolbar and select "solve")

\[ -x^2 + y = 5 \cdot x - 6 \]

3.(b,e): Plot the following functions in Cartesian coordinates

\[ y = 3x - 9 \]

\[ PV = k \quad k \text{ is a constant} \]
Find the roots by the graph method and then by calculation

\[ y_1 = x^2 - 6x + 3 \]

Find the roots using the graph and the "root" function

**Problem 2**

*(2-1 (i,j), Barrantes)* Determine the roots of the following equations, and then state whether in each case the roots are the zeros of the function

2-1(i) \[ x^2 + y^2 = 4 \]

2-1(j) \[ (x - 2)^2 + (y + 4)^2 = 9 \]

2-2 *(b,e), Barrantes* Plot the following functions in plane polar coordinates from 0 to \(2\pi\)

2-2 (b) \[ r = 4 \cdot \sin \theta \]

2-2 (e) \[ r = 4 \cdot \sin \theta \cos \theta \]

**II. Partial Derivatives (Barrantes--Chap. 4*)

**Problem 3 -- 4-1 (n.q)**

4.1*(n, q)* Differentiate the following functions. (All upper case letters are constants.)

\[ c = \frac{E^2}{A} \left( z^2 - \frac{27}{8} \cdot z \right) \]

"Use <cntrl>= in equation. Set cursor at z, pull down **Symbolics** menu, select differentiate on variable."

Then solve for z using again the **Symbolics** menu; select variable and solve with the editing cursor on z

\[ f(z) = \frac{E^2}{A} \left( z^2 - \frac{27}{8} \cdot z \right) \]

"Repeat using new function that explicitly depends on z."

\[ d = 3 \cdot \cos \left( \theta \right)^2 - 1 \]

Then solve for \(\theta\) using again the **Symbolics** menu; select variable and solve with the editing cursor on \(\theta\)
III. Integral Calculus (Barrante--Chap. 5)

Two major approaches to integral calculus.

1. Consider the integral as an antiderivative (inverse of differentiation).

\[ y = f(x) \text{ therefore } \frac{dy}{dx} = \frac{df(x)}{dx} = f(x)' = y' \]

\[ dy = f(x)' \, dx \quad \text{(differential form)} \]

_The question to pose is what function \( f(x) \) when differentiated gives \( f(x)' \) the first derivative?_  
_The function \( f(x) \) is called the integral of the differential and is noted_

\[ f(x) = \int f(x)' \, dx \]

2. Consider the integral as the sum of many similar, infinitesimal elements (area under the curve). This assigns a physical meaning to the integral.

For a more complete treatment of Integral Calculus you are encouraged to read Barrante Chapter 5

Problem 4 -- 5-1(a), 5-2 (a) and 5-3 (h)

5-1(a) \[ \int 5 \cdot x^3 \, dx \]

5-2 (a) \[ \int e^{-4x} \, dx \]

5-3 (h) for a equals 2, 5 \[ \int_0^\infty x^2 \cdot e^{-(a \cdot x)^2} \, dx \]

IV. Matrices and Determinants (Barrante--Chap 9)

A matrix is defined as a 2-D array of numbers. It can contained an equal numbers of rows and columns (rectangle) or different number of rows and columns and it is called an _mxn matrix_. The simplest matrix is a column or a row matrix with one column or one row.

\[ a := 1 \quad b := 2 \quad c := 3 \quad d := 4 \]

\[ M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

Determinant is obtained by \( a \times d - b \times c \) where a, b, c and d are the elements of the matrix.
determinant: \( ad - b\cdot c = -2 \)  \( \text{MathCAD built in function} \quad |M| = -2 \)

Matrix Algebra

Interchanging rows and columns does not change the determinant of the matrix. Interchanging any 2 rows or columns will change the sign of the determinant. The determinant of a square matrix is zero if any 2 rows or 2 columns are identical. The determinant is multiplied by \( k \) if any row or column is multiplied by a number \( k \). Two matrices are added by adding their elements. Two matrices are multiplied as shown below

\[
\begin{align*}
M_1 &= \begin{pmatrix}
a_1 & a_2 \\
b_1 & b_2 \\
\end{pmatrix} & M_2 &= \begin{pmatrix}
c_1 & c_2 \\
d_1 & d_2 \\
\end{pmatrix} \\
M_1 + M_2 &= \begin{pmatrix}
a_1 + c_1 & a_2 + c_2 \\
b_1 + d_1 & b_2 + d_2 \\
\end{pmatrix} \\
M_1 \cdot M_2 &= \begin{pmatrix}
a_1\cdot c_1 + a_2\cdot d_1 & a_1\cdot c_2 + a_2\cdot d_2 \\
b_1\cdot c_1 + b_2\cdot d_1 & b_1\cdot c_2 + b_2\cdot d_2 \\
\end{pmatrix}
\end{align*}
\]

Solving linear equations

Example

\[
\begin{align*}
x + y + z &= 2 \\
3\cdot x + y - 2\cdot z &= -5 \\
2\cdot x - y - 3\cdot z &= -5 \\
\end{align*}
\]

This set of equation can also be represented by the product of 2 matrices

\[
M_3 := \begin{pmatrix}
1 & 1 & 1 \\
3 & 1 & -2 \\
2 & -1 & -3 \\
\end{pmatrix} \quad \text{coefficient of x, y and z} \quad M_4 = \begin{pmatrix}
x \\
y \\
z \\
\end{pmatrix} \quad \text{product} \quad M_3 \cdot M_4 = \begin{pmatrix}
2 \\
-5 \\
-5 \\
\end{pmatrix}
\]

a) calculate the determinant of the coefficient \( D := |M_3| \)

\( D = -5 \)

b) calculate the determinant of the \( M_3 \) after substituing the first column with 2, -5, -5

\[
M_3' := \begin{pmatrix}
2 & 1 & 1 \\
-5 & 1 & -2 \\
-5 & -1 & -3 \\
\end{pmatrix} \quad D_1 := |M_3'| \quad D_1 = -5
\]

c) calculate the determinant of the \( M_3 \) after substituing the second column with 2, -5, -5
M''_3 := \begin{pmatrix} 1 & 2 & 1 \\ 3 & -5 & -2 \\ 2 & -5 & -3 \end{pmatrix} \quad D2 := \det(M''_3) \quad D2 = 10

d) calculate the determinant of the M'_3 after substituing the third column with 2, -5, -5

M'''_3 := \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & -5 \\ 2 & -1 & -5 \end{pmatrix} \quad D3 := \det(M'''_3) \quad D3 = -15

Finally

\begin{align*}
x &:= \frac{D1}{D} \\
y &:= \frac{D2}{D} \\
z &:= \frac{D3}{D}
\end{align*}

x = 1 \quad y = -2 \quad z = 3

Problem 6

Matrix algebra. Given the following matrices:

\[
A := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 16 \\ 3 & 9 & 27 & 81 \\ 4 & 16 & 64 & 256 \end{pmatrix}, \quad B(x) := \begin{pmatrix} x & 1 & 0 & -1/2 \\ 1 & x & 0 & 1 \\ 0 & 1 & x & 1 \\ 1/2 & 0 & 1 & x \end{pmatrix}, \quad z := \begin{pmatrix} r \\ s \\ t \\ u \end{pmatrix}
\]

(a) Compute the determinant \( \det(A) \).

(b) Find the roots of the polynomial equation \( \det(B(x)) = 0 \).

(c) Solve for the components of z when \( A \cdot z = \begin{pmatrix} -4 \\ -2 \\ 0 \\ 2 \end{pmatrix} \).

V. Differential Equations (Barrante--Chap 7)

Format required to solve a differential equation or a system of differential equations using one of the command-line differential equation solvers such as \texttt{rkfixed}, \texttt{Rkadapt}, \texttt{Radau}, \texttt{Stifb}, \texttt{Stifr} or \texttt{Bulstoer}.

For a numerical routine to solve a differential equation (DE), we must somehow pass the differential equation as an argument to the solver routine. A standard form for all DEs will allow us to do this.
Ordinary differential equations (first order). To solve this type of DE the solving block shown below should be used (Given and Odesolve)

**Given**

\[ y(x) = f(x) \]  \quad \text{initial condition } y(x_0) = 1 \quad \text{for instance}

\[ y := \text{Odesolve} \left( x, x_{\text{end}} \right) \]

plot the solution

**The idea is to change the n-th order ODE into a system of n coupled first-order differential equations.**

**Basic idea:** get rid of any second, third, fourth, etc. derivatives that appear, leaving only first derivatives.

**Example:**

\[
\frac{d^3 y}{dx^3} - \left( \frac{d}{dx} \right)^2 \left( x^2 - 1 \right) - 2 \cdot x \cdot y - e^{-500 \cdot x^2} = 0
\]

This is third order, has non constant coefficients and is neither linear nor homogeneous. To solve it we will introduce 3 new functions:

\[ A(x) = y, \quad B(x) = y', \quad C(x) = y''. \]

Then the derivatives of A, B, C are simply given by

\[
\frac{d}{dx} A = B, \quad \frac{d}{dx} B = C.
\]

And the derivative of C can be found by solving the given differential equation:

\[
\frac{d}{dx} C = 2 \cdot x \cdot A + \left( x^2 - 1 \right) \cdot B^2 + e^{-500 \cdot x^2}
\]

Suppose the initial conditions are: \( y(0) = 1, \ y'(0) = 0, \ y''(0) = -1 \). We will find and plot the values of \( y(x) \) in the interval \( 0 \leq x \leq 4 \).

\[ x := \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad D(x, y) := \begin{bmatrix} y_1 \\ y_2 \\ 2 \cdot x \cdot y_0 + \left( x^2 - 1 \right) \cdot (y_1)^2 + e^{-500 \cdot x^2} \end{bmatrix} \]

\[ r := 2000 \quad \text{2000 steps} \]

\[ x_0 := 0 \quad x_1 := 4 \quad \text{the interval} \]

The integration is defined as follows: 5 arguments, \( y \) which is a vector (here a 3x1 matrix) with the BC of the solution and the first and second derivative at the initial value, then \( x_0 \) and \( x_1 \) which represent the initial and final values of \( x \), \( r \) the number of points between initial and final values and finally the vector \( D \) that contains the first and the second derivative of the solution.
\( i := 0 \ldots r \) \hspace{1cm} \textit{index for steps}

\( Q := \text{rkfixed}(y,x_0,x_1,r,D) \) \hspace{1cm} \textit{Runge-Kutta integration}

Plot the solution \hspace{1cm} its derivative \hspace{1cm} its second derivative

\[
\begin{align*}
\text{Problem 5 (7-1 (a, c, e))} \\
(\text{a}) \quad \frac{dy}{dx} + 3 \cdot y &= 0 \\
\text{interval } 0<u<10 \quad \text{nstep} := 100 \\
\text{initial conditions } y(0) = 1 \\
\end{align*}
\]

\[
\begin{align*}
(\text{c}) \quad \frac{d^2y}{dx^2} + \left( \frac{dy}{dx} \right) + y &= 0 \\
\text{interval } 0<u<10 \quad n := 1000 \quad \text{number of steps} \\
\text{initial conditions } y(0) = 1, \ y'(0) = 1 \text{ and } y''(0) = 0 \\
\end{align*}
\]

\[
\begin{align*}
(\text{e}) \quad \frac{d^2y}{dx^2} - 9 \cdot y &= 0 \\
\text{interval } 0<u<5 \quad n := 500 \\
\text{initial conditions } y(0) = -1, \ y'(0) = 0 \text{ and } y''(0) = 1 \\
\end{align*}
\]