

# Coherence statistic for detecting gravitational-wave signals from inspiraling compact binaries with laser-interferometers

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## Abstract

We describe a coherence statistic for the detection of gravitational waves from inspiraling compact binaries with a network of laser-interferometric detectors having arbitrary orientations and arbitrary locations around the globe. Its derivation is based on the maximum-likelihood method (MLM) and it is optimal for a search in stationary, Gaussian noise. We present the exact form of this statistic for non-coincident multi-detector networks comprising of the LIGOs at Hanford and Livingston as well as GEO in Hannover.

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# I. INTRODUCTION

We present here a brief derivation of and introduction to the coherence statistic that is discussed in more detail in Refs. [1,2]. This is a statistic that can prove useful in performing a search for inspiral signals in the data from the coincident engineering runs of the LIGOs and GEO. The main motivation behind composing this draft is to apprise the members of the Inspiral Upper Limit (IUL) group about this statistic. We also describe towards the end the role played by the Inspiral Search Code (ISC), which is being developed by a task group within the IUL group, in implementing a multi-detector search based on the coherence statistic. More importantly, we point out the additional codes that the Multiple-Interferometer (MI) task group is developing in order to complement the ISC in executing such a search.

## II. MATHEMATICAL FRAMEWORK

### A. Reference Frames

Our first aim is to obtain a quantity that defines the *response* of an arbitrary network of broadband detectors to an incoming gravitational wave. In this quest, it is necessary to understand how the responses of arbitrarily oriented and arbitrarily located individual detectors to such a wave relate to one another. This is aided by introducing the three different frames of reference that arise naturally in such a problem, namely, (i) the wave frame, (ii) the network frame, and (iii) the frame of a representative detector in the network. We define these reference frames in terms of the following right-handed, orthogonal, three-dimensional Cartesian coordinates:

(i) *Wave frame*: We associate with this frame the coordinates  $(X, Y, Z)$ . The gravitational wave, which is assumed to be weak and planar, is taken to travel along the positive  $Z$ -direction; then  $X$  and  $Y$  denote the axes of the polarization ellipse of the wave.

(ii) *Network frame*: There is no unique definition for this frame. For Earth-based detectors being discussed here, if the network has a large number of detectors (say,  $M > 3$ ), a convenient choice is a frame attached to the center of the Earth. Let the coordinate system that defines this frame be  $(x_E, y_E, z_E)$ . The  $x_E$  axis lies along the line joining Earth's center and the equatorial point that lies on the meridian passing through Greenwich, England. It points radially outwards. The  $z_E$  axis is chosen to lie in the direction of the line passing through the center of Earth and the north pole. The  $y_E$  axis is chosen to form a right-handed coordinate system with the  $x_E$  and  $z_E$  axes.

For a network consisting of  $M \leq 3$  detectors, certain calculations can be simplified by using the fact that the corner stations (or hubs) of all the detectors will lie on a single plane. Specifically, for  $M = 3$  we define the network frame such that one of the detectors is at its origin, a second detector is on one of the coordinate axes, say,  $z$ , and the third lies on one of the coordinate planes containing the  $z$  axis, say, the  $x - z$  plane.

(iii) *Detector frame*: Let  $(x_{(I)}, y_{(I)}, z_{(I)})$  (with  $I = 1, 2, \dots, M$ ) denote the orthogonal coordinate frame attached to the detector labeled  $I$ . The  $(x_{(I)}, y_{(I)})$  plane contains the detector arms and is assumed to be tangent to the surface of the Earth. The  $x_{(I)}$  axis bisects the angle

between the detector’s arms. The direction of the  $y_{(I)}$  axis is chosen in such a way that  $(x_{(I)}, y_{(I)}, z_{(I)})$  form a right-handed coordinate system with the  $z_{(I)}$  axis pointing radially out of Earth’s surface.

Apart from the above choices for frames, we define a fourth frame, namely, the frame of a “fiducial” detector (henceforth referred to as the “fide”). This frame serves as a reference with respect to which the locations or orientations of each of the detectors in a network shall be specified. Indeed, we will develop our whole formalism for a general network using the fide frame as a reference. When one considers specific cases of networks, it may prove useful to identify the fide frame with one of the three frames defined above, depending upon suitability.

Physical quantities in these frames are related by orthogonal transformations that rotate one frame into another. These orthogonal transformations are defined in terms of three sets of Euler angles that specify the orientation of one frame with respect to another. To understand these relations, let  $\mathcal{O}$  be an orthogonal transformation matrix. Let  $(\phi, \theta, \psi)$  be the Euler angles through which one must rotate the fide frame to the align with the wave frame. Then

$$\mathbf{x}_{\text{wave}} = \mathcal{O}(\phi, \theta, \psi)\mathbf{x}_{\text{fide}} \quad , \quad (2.1)$$

where  $\mathbf{x}_{\text{fide}}$  denote the axes of the fide frame and  $\mathbf{x}_{\text{wave}}$ , those of the wave frame.  $\mathcal{O}$  is an orthogonal matrix as defined in Ref. [3]. Similarly, if  $\boldsymbol{\alpha}_{(I)} := (\alpha_{(I)}, \beta_{(I)}, \gamma_{(I)})$  are the Euler angles that rotate the fide frame to the frame of the  $I$ -th detector, then

$$\mathbf{x}_{\text{detector}_{(I)}} = \mathcal{O}(\alpha_{(I)}, \beta_{(I)}, \gamma_{(I)})\mathbf{x}_{\text{fide}} \quad , \quad (2.2)$$

where  $\mathbf{x}_{\text{detector}_{(I)}}$  denote the axes of the  $I$ -th detector frame.

We use the following convention for symbols in this paper, unless otherwise specified. When it is useful to keep track of the complex nature of a network-based or individual detector-based *variable* we denote it by an uppercase Roman letter, whereas the lower case letters are reserved for real variables.<sup>1</sup> Network-based vectors are displayed in the Sans Serif font. The *label*  $I$  in the superscript or subscript of a variable denotes a (real) natural number that associates it with a particular detector. It ranges from 1 to  $M$ , where  $M$  is the total number of detectors in a network. It can be considered as a vector index over detectors. We use the index  $I$  for several of the network variables. However, certain quantities that do not obviously display a vector character, but still pertain to a detector are denoted by enclosing the index in parentheses, such a  $(I)$ . Einstein summation convention over repeated indices is used for brevity, unless explicitly stated.

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<sup>1</sup>Note that quantities such as the gravitational constant,  $G$ , though written in upper case, are not complex since they do not represent any inherent characteristic of the network or an individual detector. On the other hand, we shall not use an uppercase letter to denote a complex quantity when its complex nature is apparent from other means, such as by the use of a tilde, e.g., in  $\tilde{n}$ , which denotes the, in general, complex Fourier transform of the real quantity  $n$ .

## B. Wave tensor, detector tensor, and beam-pattern functions

A gravitational wave can be represented by metric tensor fluctuation,  $h_{\mu\nu}$ , about the background space-time which we take to be flat. The subscripts  $\mu$  and  $\nu$  denote space-time indices. In the transverse trace-free (TT) gauge, the non-vanishing components of  $h_{\mu\nu}$  in the wave frame are  $h_{xx} = -h_{yy} \equiv h_+$ ,  $h_{xy} = h_{yx} \equiv h_\times$ . Here,  $h_+$  and  $h_\times$  are the two linear-polarization components of the wave. When a metric fluctuation specifically represents a gravitational wave, its spatial part is identified as twice the wave tensor,  $w_{ij}$ , where  $i$  and  $j$  refer to space indices and take values 1, 2, and 3 (see Ref. [4]). In the TT gauge, the wave tensor is a symmetric trace-free (STF) tensor of rank 2 [1]. In any arbitrary frame, the wave tensor can be expressed in terms of its circular-polarization components as,

$$w^{ij}(t) = \frac{1}{2} \left[ (h_+(t) + ih_\times(t)) e_R^{ij} + (h_+(t) - ih_\times(t)) e_L^{ij} \right] , \quad (2.3)$$

where  $e_{R,L}^{ij}$  are the right and left-circular polarization tensors, respectively. The  $e_{R,L}^{ij}$  are both second rank STF tensors and obey the orthonormality conditions,

$$e_{L,R}^{ij} e_{L,R}^*{}_{ij} = 1 , \quad e_{L,R}^{ij} e_{R,L}^*{}_{ij} = 0 , \quad (2.4)$$

where a star denotes complex conjugation. The reality of the wave tensor ensures that  $e_L^{ij*} = e_R^{ij}$ . Thus, the wave-tensor expression (2.3) simplifies to

$$w^{ij}(t) = \Re \left[ (h_+(t) + ih_\times(t)) e_R^{ij} \right] , \quad (2.5)$$

where  $\Re[A]$  denotes the real part of a complex quantity  $A$ .

In an arbitrary reference frame, the polarization tensor can be expressed as

$$e_L^{ij} = m^i m^j , \quad (2.6)$$

where the  $m^k$  is the  $k$ -th component of a complex null vector  $\mathbf{m}$  in that reference frame. It is defined as

$$m^k = \frac{1}{\sqrt{2}} (e_X^k + i e_Y^k) , \quad (2.7)$$

where  $\mathbf{e}_X$  and  $\mathbf{e}_Y$  are unit vectors in the  $X$  and  $Y$  direction of the wave axes, respectively [4].

The  $I$ -th detector tensor,  $d_{ij}^I$ , is given by

$$d_{ij}^I = n_{(I)1i} n_{(I)1j} - n_{(I)2i} n_{(I)2j} , \quad (2.8)$$

where  $\mathbf{n}_{(I)1}$  and  $\mathbf{n}_{(I)2}$  are the unit vectors along the arms of the  $I$ -th interferometer, which may not have orthogonal arms.

When detectors are distributed around the globe there are, in general, relative delays in the arrival times of a particular phase of a given wave at different locations. Let  $\tau_{(I)}(\theta, \phi)$  be the relative delay between the arrival times at the  $I$ -th detector and the fide, where the

source direction is given by  $(\theta, \phi)$ . If  $\hat{\mathbf{n}}(\theta, \phi)$  is the unit vector along the direction of the wave, i.e.,  $\hat{\mathbf{n}}(\theta, \phi) = \hat{Z}$ , then

$$\tau_{(I)}(\theta, \phi) = \frac{(\mathbf{r}_{(I)} - \mathbf{r}_{(f)}) \cdot \hat{\mathbf{n}}(\theta, \phi)}{c} , \quad (2.9)$$

where  $\mathbf{r}_{(I)}$  and  $\mathbf{r}_{(f)}$  are the position vectors of the  $I$ -th detector and fide, respectively, with respect to any given reference frame. Note that  $\tau_{(I)}(\theta, \phi)$  can take positive as well as negative values.

The signal in the  $I$ -th detector is the scalar

$$s^I(t) = w^{ij}(t - \tau_{(I)})d_{ij}^I , \quad (2.10)$$

which, by definition, is invariant under coordinate transformations. Above,  $w^{ij}(t)$  is the wave tensor at the location of the fide at time  $t$ . It is a function of  $h_+(t)$  and  $h_\times(t)$ , which define the amplitudes of the two polarization components at time  $t$  and at the location of the fide. The above definition shows that the signal depends on the projections of the polarization tensors,  $e_{L,R}^{ij}$ , onto the  $I$ -th detector tensor,  $d_{ij}^I$ . These projections are

$$F^I = e_L^{ij}d_{ij}^I, \quad F^{I*} = e_R^{ij}d_{ij}^I , \quad (2.11)$$

which are the beam-pattern functions for the left- and right-circular polarizations, respectively. They depend on the direction of the source, the orientation and the arm-opening angle of the detector. Owing to any motion of the source with respect to the detector this orientation can change with time. Hence, in general,  $F^I$  are functions of time. Since we will be concerned here with only short-duration signals, we will assume these functions to be independent of time (which is valid to a very good approximation). Using the above definition of the beam-pattern functions and the wave-tensor expression (2.5) in Eq. (2.10), we find the signal to be

$$s^I(t) = \Re \left[ \left( h_+^I(t) + ih_\times^I(t) \right) F^{I*} \right] , \quad (2.12)$$

where  $h_+^I(t) \equiv h_+(t - \tau_{(I)})$  and  $h_\times^I(t) \equiv h_\times(t - \tau_{(I)})$  are the time-delayed amplitudes of the two polarizations of the wave at detector  $I$ .

### C. Network signal and network statistic

The signal from an inspiraling binary will typically not stand above the broadband noise of the interferometric detectors; the concept of an absolutely certain detection does not exist in such a case. Only probabilities can be assigned to the presence of an expected signal. In the absence of prior probabilities, such a situation demands a decision strategy that maximizes the detection probability for a given false alarm probability. This is termed as the Neyman-Pearson criterion [5]. Such a criterion implies that the decision must be based on a statistic called the likelihood ratio (LR). It is defined as the ratio of the probability that a signal is present in an observation to the probability that it is not. This is the criterion we employ in formulating our detection strategy.

In order to define a strategy to search for signals in a noisy environment, it is important to recognize the characteristics of the noise. Here, we assume that the noise,  $n^I(t)$ , in the  $I$ -th detector (a) has a zero mean and (b) is mostly stationary<sup>2</sup> and statistically as well as algebraically independent of the noise in any other detector. These requirements are mathematically summarized, respectively, as:

$$\overline{n^I(t)} = 0, \quad (2.13a)$$

$$\overline{\tilde{n}_I^*(f)\tilde{n}^J(f')} = s_{h(I)}(f)\delta(f-f')\delta_I^J, \quad (2.13b)$$

where the over-bar on a symbol denotes the ensemble average of that quantity and the tilde denotes the Fourier transform,

$$\tilde{n}^I(f) = \int_{-\infty}^{\infty} n^I(t)e^{-2\pi ift} dt. \quad (2.14)$$

Also,  $s_{h(I)}(f)$  is the one sided power-spectral-density (PSD) of the  $I$ -th detector. Note that  $s_{h(I)}(f)$  is the Fourier transform of the auto-covariance of the noise in detector  $I$ . We also assume the noise to be additive. This implies that when a signal is present in the data, then  $x^I(t)$  is given by

$$x^I(t) = s^I(t) + n^I(t), \quad (2.15)$$

otherwise  $x^I(t) = n^I(t)$ . Finally, we assume that the noises are Gaussian, i.e., the two moments in Eq. (2.13) are sufficient to completely characterize the noises statistically.

As we shall see below, an important tool in the theory of detection of known signals in noisy environments is the cross correlation between a signal template and a detector's output. In order to define it, consider two real, sufficiently smooth, and absolutely integrable functions of time, namely,  $a(t)$  and  $b(t)$ . For the purposes of this paper, we can assume that the signal template,  $s^I(t)$ , and the detector outputs,  $x^I(t)$ , belong to this category of functions to a good approximation. A cross correlation can be represented in terms of an inner product, which is defined as

$$\langle a, b \rangle_{(I)} = 2\Re \int_0^\infty df \frac{\tilde{a}^*(f)\tilde{b}(f)}{s_{h(I)}(f)}, \quad (2.16)$$

where  $\tilde{a}(f)$  and  $\tilde{b}(f)$  are the Fourier transforms of  $a(t)$  and  $b(t)$ , respectively. In order to obtain the correlation between a complex function  $A(t) = a_1(t) + ia_2(t)$  and a real function  $b(t)$ , we adopt the following convention to define the inner product

$$\langle A, b \rangle \equiv \langle a_1, b \rangle - i\langle a_2, b \rangle. \quad (2.17)$$

This definition is consistent with the convention of (2.16) where the complex conjugation is performed on the first entry in the inner product.

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<sup>2</sup>In reality, detector noise contains non-Gaussian and non-stationary components. Such features can be accommodated in our treatment, by using vetoing techniques of the kind described in Ref. [6].

For a network of  $M$  detectors, the data consist of  $M$  data trains,  $\{x^I(t) | I = 1, 2, \dots, M$  and  $t \in [0, T]\}$ . The network matched-template can be obtained naturally by the maximum-likelihood method, where the decision whether the signal is present or not is made by evaluating the likelihood ratio (LR) for the network [5]. Under the assumptions made on the noise, the network LR, denoted by  $\lambda$ , is just a product of the individual detector LRs. In addition, for Gaussian noise, the logarithmic likelihood ratio (LLR) for the network is just the sum of the LLRs of the individual detectors [7],

$$\ln \lambda = \sum_{I=1}^M \ln \lambda_{(I)} \quad , \quad (2.18)$$

where

$$\ln \lambda_{(I)} = \langle s^I, x^I \rangle_{(I)} - \frac{1}{2} \langle s^I, s^I \rangle_{(I)} . \quad (2.19)$$

The network LLR takes a compact form in terms of the network inner-product,

$$\langle \mathbf{s}, \mathbf{x} \rangle_{NW} = \sum_{I=1}^M \langle s^I(t), x^I(t) \rangle_{(I)} \quad , \quad (2.20)$$

where

$$\mathbf{s}(t) = (s^1(t), s^2(t), \dots, s^M(t)) \quad (2.21)$$

is the network template-vector, which comprises of individual detector-templates as its components, and

$$\mathbf{x}(t) = (x^1(t), x^2(t), \dots, x^M(t)) \quad (2.22)$$

is the network data-vector.

It can be shown by using the Schwarz inequality that the network template,  $\mathbf{s}$ , defined above yields the maximum signal-to-noise (SNR) amongst all linear templates and, hence, is the matched template. As shown in Ref. [7], in terms of the above definitions, the network LLR takes the following simple form:

$$\ln \lambda = \langle \mathbf{s}, \mathbf{x} \rangle_{NW} - \frac{1}{2} \langle \mathbf{s}, \mathbf{s} \rangle_{NW} \quad , \quad (2.23)$$

which is a function of the source parameters that determine  $\mathbf{s}$ . Given  $\mathbf{s}$ , different selections of source-parameter values and, therefore, different values of  $\mathbf{s}$  result in varying magnitudes of the LLR. The selection that gives the maximum value stands the best chance for beating the pre-set threshold on the LLR. Since scanning the complete source-parameter manifold for the maximum of LLR is computationally very expensive, we propose to perform its maximization analytically over as many parameters as possible. This requires the knowledge of the analytic dependence of the network matched-template on source parameters. This is what we seek below.

### III. THE SIGNAL

The Inspiral Search Code, in its present form, has the functionality to search in the data from a single detector for the 2PN waveforms of inspiraling, non-spinning compact binaries. It does this by evaluating the detection statistic (and the  $\chi^2$  statistic) and comparing them with pre-set thresholds. It might also be possible to extend this code to search for other waveforms, such as the ones corresponding to the 2.5 PN  $P$  approximants. In any case, the ISC depends on the existence of a bank of templates for the corresponding waveform. Alternatively, one can implement the Fast Chirp Transform (FCT) code to evaluate the detection statistic. In either case, one needs to know beforehand the form of the signal in a given detector, and how it relates to the response of other detectors to the same source. The form of the network detection statistic given in the next section basically remains unaffected by the order of the PN approximation to which the waveforms are computed, as long as we use the *restricted* waveforms. Therefore, for simplicity, we shall use the chirp expression obtained in the Newtonian, quadrupole approximation to express the detector responses.

Assume that the binary is at a luminosity distance of  $r$  from the Earth<sup>3</sup>. Further, let  $m_1$  and  $m_2$  be the masses of the individual stars. Then, in the Newtonian approximation the two corresponding GW linear-polarization components in the wave frame, at the location of the fide, are

$$h_+(t; r, \delta_c, t_c, \xi) = \frac{2\mathcal{N}}{r} a^{-1/4}(t; t_c, \xi) \frac{1 + \cos^2 \epsilon}{2} \cos[\chi(t; t_c, \xi) + \delta_c] , \quad (3.1a)$$

$$h_\times(t; r, \delta_c, t_c, \xi) = \frac{2\mathcal{N}}{r} a^{-1/4}(t; t_c, \xi) \cos \epsilon \sin[\chi(t; t_c, \xi) + \delta_c] , \quad (3.1b)$$

where

$$\mathcal{N} \equiv \left[ \frac{2G^{5/3} \mathcal{M}^{5/3} (\pi f_s)^{2/3}}{c^4} \right] \quad (3.2)$$

is a constant appearing in the chirp amplitude having the dimensions of length. It depends on the binary's ‘chirp’ mass,  $\mathcal{M} \equiv (m_1 m_2)^{3/5} / (m_1 + m_2)^{1/5}$ , and a fiducial chirp frequency,  $f_s$ . Usually,  $f_s$  is taken to be the lowest frequency in the bandwidth of a detector - the seismic cut-off - hence the reason for the subscript  $s$ . This choice of the fiducial frequency maximizes the duration of tracking the chirp because the chirp frequency increases monotonically with time.

A quantity closely related to the chirp mass is the so-called chirp time,

$$\xi = 34.5 \left( \frac{\mathcal{M}}{M_\odot} \right)^{-5/3} \left( \frac{f_s}{40 \text{ Hz}} \right)^{-8/3} \text{ sec.} , \quad (3.3)$$

which equals the time duration for which the chirp exists in a detector's sensitivity window from the time of arrival until the time of final coalescence. The time of arrival,  $t_a$ , is defined

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<sup>3</sup>Here,  $r$  is not to be confused with the magnitude of a detector position-vector, which always carries as an index the label of the detector, i.e.,  $(I)$  or  $(f)$ .

as the time when the instantaneous frequency of the chirp equals the fiducial frequency, i.e.,  $f(t_a) = f_s$ . Formally, the coalescence time,  $t_c$ , is the time at which the chirp frequency diverges (see Eq. (3.5)). The corresponding phase of the wave-form at  $t_c$  is  $\delta_c$ . We define the quantity

$$a(t; t_c, \xi) = \frac{t_c - t}{\xi}, \quad (3.4)$$

and the instantaneous frequency,

$$f(t; f_s, t_c, \xi) = f_s a^{-3/8}(t; t_c, \xi) = f_s \left( \frac{t_c - t}{\xi} \right)^{-3/8}, \quad (3.5)$$

which diverges at final coalescence. The above expression also shows that

$$t_c = t_a + \xi. \quad (3.6)$$

Finally, the instantaneous phase of the waveform is  $\chi(t) + \delta_c$ , where

$$\chi(t; f_s, t_c, \xi) \equiv -2\pi \int_t^{t_c} f(t'; f_s, t_c, \xi) dt' = -\frac{16}{5}\pi f_s \xi a^{5/8}(t; t_c, \xi). \quad (3.7)$$

The two GW polarization amplitudes at the  $I$ -th detector site are obtained by substituting  $t$  with  $(t - \tau_{(I)})$  in Eqs. (3.1), (3.4), (3.5), and (3.7).

A chirp signal registers itself in a detector's output only after its instantaneous frequency crosses the seismic cut-off of that detector. Thus, a signal arrives in the  $I$ -th detector's bandwidth when its instantaneous frequency reaches  $f = f_{s(I)}$  and it lasts there for a time duration equaling  $\xi_{(I)} = \xi \left( f_{s(I)}/f_s \right)^{-8/3}$ . Alternatively put, the chirp waveform at the  $I$ -th detector starts at  $t = t_c + \tau_{(I)} - \xi_{(I)}$  and ends at  $t = t_c + \tau_{(I)}$ .

In order to formulate a strategy for detecting a chirp, it helps to isolate the factors in the two polarizations,  $h_{+, \times}$ , that are time dependent from those that are not. To this end, we define two mutually orthogonal normalized waveforms  $s_0^I$  and  $s_{\pi/2}^I$ , with  $s_{0, \pi/2}^I(t) = s_{0, \pi/2}(t - \tau_{(I)})$ . Their complex combination,  $S^I := s_0^I + i s_{\pi/2}^I$ , is a normalized complex signal; it has the simple form:

$$S^I(t; t_c, \xi) \equiv \frac{a^{-1/4}(t - \tau_{(I)}; \xi)}{g_{(I)} \sqrt{\xi}} e^{i\chi(t - \tau_{(I)}; \xi)}. \quad (3.8)$$

Here,  $g_{(I)}$  is a normalization factor such that

$$\langle S^I, S^I \rangle_{(I)} = 1. \quad (3.9)$$

We now obtain an expression for the normalization factor,  $g_{(I)}$ . In the stationary-phase approximation (SPA), the Fourier transform of  $S^I(t)$  for positive frequencies is,

$$\begin{aligned} \tilde{S}^I(f; t_c, \xi) &= \int_{-\infty}^{\infty} S^I(t; t_c, \xi) e^{-2\pi i f t} dt \\ &= \frac{2}{g_{(I)}} \sqrt{\frac{2}{3f_s}} \left( \frac{f}{f_s} \right)^{-7/6} \exp \left[ i\Psi_{(I)}(f; f_s, t_c, \xi) \right], \end{aligned} \quad (3.10)$$

where

$$\begin{aligned}\Psi_{(I)}(f; f_s, t_c, \xi) &= -2\pi f_s \left[ \frac{f}{f_s} t_c + \frac{f}{f_s} \tau_{(I)} + \frac{3}{5} \xi \left( \frac{f}{f_s} \right)^{-5/3} \right] + \frac{\pi}{4} \\ &\equiv \Psi(f; f_s, t_c, \xi) - 2\pi f \tau_{(I)} \quad ,\end{aligned}\tag{3.11}$$

for the Newtonian chirp.<sup>4</sup> Note that  $\Psi_{(I)} = \Psi$  for vanishing time-delay ( $\tau_{(I)} = 0$ ). Thus,  $\Psi$  defines the phase in the Fourier transform of the normalized complex signal at the fide, in the SPA. The normalization condition (3.9) implies that,

$$g_{(I)}^2 = \frac{8}{3} f_s^{4/3} \int_{f_{s(I)}}^{\infty} \frac{df}{f^{7/3} s_{h(I)}(f)} \quad ,\tag{3.12}$$

where  $f_{s(I)}$  is the seismic cut-off for the  $I$ -th detector.

### A. The signal at a detector

The signal due to a Newtonian chirp at the  $I$ -th detector can be expressed in terms of  $s^I(t)$ . It is obtained by the chirp-specific components  $h_{+,x}^I$  from (3.1). We now express the GW circular-polarization components,  $(h_+^I + i h_x^I)$ , in terms of the normalized complex signal  $S^I$  and the overall amplitude  $\kappa$ . In the special case of the face-on binary (i.e.,  $\epsilon = 0$ ), the signal at the detector is given by

$$s^I(t) = 2\kappa \Re \left[ g_{(I)} F_I^* S^I(t) e^{i\delta_c} \right] \quad ,\tag{3.13}$$

where  $\kappa \equiv \mathcal{N} \sqrt{\xi}/r$ . Note that  $\delta_c$  is detector independent and separates out as a phase factor in the expression for the complexified  $s^I(t)$ .

The generalization of (3.13) for arbitrary value of  $\epsilon$  is straightforward. In this case,  $s^I$  can be expressed as follows:

$$s^I(t) = 2\kappa \Re \left[ (E_I^* S^I) e^{i\delta_c} \right] \quad ,\tag{3.14}$$

where we have defined the extended beam-pattern functions

$$E^I = g_{(I)} \left[ \frac{1 + \cos^2 \epsilon}{2} \Re(F^I) + i \cos \epsilon \Im(F^I) \right] \quad .\tag{3.15}$$

Here,  $\Re(F^I)$  and  $\Im(F^I)$  are the real and imaginary parts of the detector beam pattern functions, respectively. In the limit  $\epsilon \rightarrow 0$ , the signal in Eq. (3.14) reduces to that in (3.13).  $E^I$  depends on the source-direction angles,  $\{\theta, \phi\}$ , the angles,  $\{\epsilon, \psi\}$ , as well as on

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<sup>4</sup>For the restricted 2PN waveform, the signal gets modified through terms appearing essentially in the phase  $\Psi(f)$ . These terms depend on powers of  $f$  that are different from those appearing in Eq. (3.11) and are parametrized by both  $\xi$  and the 1.5PN chirp time.

the orientation of the  $I$ -th detector relative to the fide, given by the Euler angles  $\boldsymbol{\alpha}_{(I)}$ . Also,  $E^I$  depends on the signal-normalization factor  $g_{(I)}$ , which expresses the sensitivity of the detector to the incoming signal. Thus, the signal at a detector depends on a total of eight independent parameters, namely,  $\{r, \delta_c, \psi, \epsilon, t_c, \xi, \theta, \phi\}$ . The ranges of the four angles are as follows:  $\psi \in [0, 2\pi]$ ,  $\epsilon \in [0, \pi]$ ,  $\phi \in [0, 2\pi]$ , and  $\theta \in [0, \pi]$ . The 2PN signal depends on an additional parameter, *viz.*, the 1.5PN chirp time.

## B. Network signal normalization

The total energy in a signal that is accessible to a network is just the network scalar  $\langle \mathbf{s}, \mathbf{s} \rangle_{NW}$ , and is given by

$$\begin{aligned} \langle \mathbf{s}, \mathbf{s} \rangle_{NW} &= \sum_{I=1}^M \langle s^I(t), s^I(t) \rangle_{(I)} \\ &= 4\kappa^2 \sum_{I=1}^M E^*_{\ I} E^I \equiv \mathbf{b}^2. \end{aligned} \quad (3.16)$$

The quantity  $\sum_{I=1}^M E^*_{\ I} E^I \equiv \mathbf{E} \cdot \mathbf{E} = \|\mathbf{E}\|^2$  is the  $\mathcal{L}^2$  norm of  $E^I$  in  $\mathcal{C}^M$ . The above analysis also suggests the normalization for the network signal. The signal vector with unit norm is defined by  $\hat{\mathbf{s}} \equiv \mathbf{s}/\mathbf{b}$ . Its components are

$$\hat{s}^I = \Re \left[ (Q_I^* S^I) e^{i\delta_c} \right] \quad , \quad (3.17)$$

where

$$Q^I \equiv \frac{E^I}{\|\mathbf{E}\|} \quad . \quad (3.18)$$

Note that the network vector  $\mathbf{Q} = (Q^1, Q^2, \dots, Q^M)$  lies in the  $M$  dimensional complex space  $\mathcal{C}^M$  and has a unit norm, *i.e.*,

$$\|\mathbf{Q}\|^2 = 1. \quad (3.19)$$

Like  $\mathbf{E}$ , even  $\mathbf{Q}$  depends on  $\{\psi, \epsilon, \theta, \phi; \boldsymbol{\alpha}_{(1)}, \dots, \boldsymbol{\alpha}_{(M)}\}$ . But for a given choice of values for the angles  $\{\theta, \phi; \boldsymbol{\alpha}_{(1)}, \dots, \boldsymbol{\alpha}_{(M)}\}$ , one can prove that  $\mathbf{Q}$  is always restricted to lie in a two-dimensional complex subspace of  $\mathcal{C}^M$  [1]. This subspace is termed as the ‘‘helicity’’ plane,  $\mathcal{H}$ . Thus, given a network of detectors (with their fixed orientations), choosing a source direction is tantamount to choosing a specific helicity plane in  $\mathcal{C}^M$ . Furthermore, by varying  $\psi$  and  $\epsilon$  one picks out different ‘‘directions’’ in this plane along which  $\mathbf{Q}$  lies.

## IV. THE NETWORK STATISTIC

In the case of a single detector, the LLR is a functional of the data as measured by that detector. For a network of  $M$  detectors, one needs to compute the statistic in terms of the network data-vector  $\mathbf{x}$ . When our assumptions about the statistical properties of detector

noise are valid, the appropriate network LLR is given by Eq. (2.23). The optimal network statistic is obtained by maximizing this LLR over the eight physical parameters that define the signal. It is this maximized LLR that must be compared with a pre-set threshold, corresponding to a given false alarm probability. In the following two subsections, we show how such a maximization over four of the parameters can be performed analytically. Further maximization over the time of final coalescence (or, analogously, over the time of arrival (at the fide)) can be performed by using FFTs in a way that is very similar to what is done in the case of a single detector [2].

We begin by analytically maximizing the network LLR with respect to two parameters that are simplest to handle, namely,  $r$  and  $\delta_c$ . Note that the network LLR obtained in Eq. (2.23) can be expressed as an explicit function of  $\mathbf{b}$ :

$$\ln \lambda = \mathbf{b} \sum_{I=1}^M \langle \hat{s}^I, x^I \rangle_{(I)} - \frac{1}{2} \mathbf{b}^2. \quad (4.1)$$

Above, the luminosity distance,  $r$ , affects only  $\mathbf{b}$ . The value of  $\mathbf{b}$  at which  $\ln \lambda$  is a maximum is  $\hat{\mathbf{b}} = \sum_{I=1}^M \langle \hat{s}^I, x^I \rangle_{(I)}$ . The hat on a parameter denotes the value at which the LLR is a maximum as a function of that parameter, keeping all other parameters fixed. Here, the value of LLR at  $\mathbf{b} = \hat{\mathbf{b}}$  is

$$\ln \lambda|_{\hat{\mathbf{b}}} = \frac{1}{2} \left( \sum_{I=1}^M \langle \hat{s}^I, x^I \rangle_{(I)} \right)^2 = \frac{1}{2} \left( \Re \left[ e^{-i\delta_c} (\mathbf{C} \cdot \mathbf{Q}) \right] \right)^2, \quad (4.2)$$

where we have defined,

$$C_I^* = c_0^I - i c_{\pi/2}^I \equiv \langle S^I, x^I \rangle_{(I)}, \quad (4.3)$$

with  $c_0^I = \langle s_0^I, x^I \rangle_{(I)}$  and  $c_{\pi/2}^I = \langle s_{\pi/2}^I, x^I \rangle_{(I)}$ .  $C^I$  is a complex quantity that combines the correlations of the two quadratures of the normalized template with the data. We proceed further and maximize the LLR in (4.2) with respect to  $\delta_c$ . This yields  $\hat{\delta}_c = \arg(\mathbf{C} \cdot \mathbf{Q})$  and the LLR maximized over  $\mathbf{b}$  and  $\delta_c$  as,

$$\ln \lambda|_{\hat{\mathbf{b}}, \hat{\delta}_c} = \frac{1}{2} |\mathbf{C} \cdot \mathbf{Q}|^2 =: \frac{1}{2} \mathbb{L}^2. \quad (4.4)$$

The maximized LLR above is a function of six parameters, namely,  $\{\epsilon, \psi, t_c, \xi, \theta, \phi\}$ .  $\mathbf{C}$  depends on  $\{t_c, \xi, \theta, \phi\}$  and  $\mathbf{Q}$  depends on  $\{\epsilon, \psi, \theta, \phi\}$ .

It is instructive at this stage to enquire what happens to the expression (4.4), which describes a network statistic, in the special case of a “network” comprising of a single detector. In such a case, we have

$$\mathbb{L}(\tau) = |C_1^*(\tau)Q^1| = |C_1(\tau)|, \quad (4.5)$$

where we used Eq. (3.19) in the last step. This is but the cross-correlation at the offset time  $\tau$  of the detector output with a chirp template. The signal-to-noise ratio is just the square of this quantity. This is indeed the detection statistic for a single detector [8]. It is analogous to the statistic “ $\rho$ ” that is computed by the Inspiral Search Code for 2PN waveforms.

We now consider a network of two identical detectors with identical noise PSDs. We make the following choice of coordinates. We select one of the two detectors to be the fide. The  $z$  axis of the fide is chosen along the line joining the two detectors. Then the second detector is taken to be located at  $(0, 0, z_2)$ , with an orientation identical to that of the fide, i.e.,  $\alpha_{(2)} = \beta_{(2)} = \gamma_{(2)} = 0$ . Owing to the same orientations, the beam-pattern functions of the two detectors are identical. If the detectors were located at the same place, then the resulting network would have mimicked a single detector, but with a higher sensitivity. The statistic given below would have been applicable for such coincident detectors (e.g., LHO4 and LLO2), but for the fact that the noises in these detectors are generally not independent. Here, however, we consider spatially separated detectors, where the relative time delay,  $\tau_{(2)}$ , provides partial information about the source-direction. The time delay is given by

$$\tau_{(2)} = z_2(\hat{\mathbf{z}} \cdot \hat{\mathbf{n}})/c \quad , \quad (4.6)$$

and the network statistic in this case is

$$\mathbb{L}(\tau) = |\mathbf{C}(\tau) \cdot \mathbf{Q}| \quad (4.7)$$

$$= \frac{1}{\sqrt{2}} |C_1^*(\tau) + C_2^*(\tau; \tau_{(2)})|. \quad (4.8)$$

Note that since the detector orientations are identical, we have  $Q^1 = Q^2$ ; then Eq. (3.19) implies that  $|Q^1| = |Q^2| = 1/\sqrt{2}$ . This means that we can gain no information about  $\epsilon$  and  $\psi$  from such a network. For a two-detector network, any given value of the time delay corresponds to more than one source directions, all of which lie on the surface of a cone whose axis coincides with the line joining the two detectors. Only when the source lies on the line passing through the two detectors is the time delay single-valued, and is of maximum magnitude for a given pair of detectors (note that we have allowed  $\tau_{(I)}$  to be negative as well). The value of the time delay  $\tau_{(2)}$  determines the opening angle of the cone. Thus, the azimuthal direction angle  $\phi$  of the wave remains undetermined in the case of two detectors. Only  $\theta$  can be estimated from the time delay that appears in the phase difference of the detector responses.  $\mathbb{L}^2$  is not the sum of the SNRs at the two detectors; instead it depends on the ‘‘coherent’’ sum of the two cross-correlations. If one were to neglect the slightly different orientations of the LLO and LHO, then the above statistic is the appropriate one for a search.

Next we consider the case of two non-coincident, identical detectors with identical noise PSDs, but with different orientations. We make the choice of coordinates identical to that given above. Since the two detectors have different orientations, the beam-pattern functions for the two detectors differ, i.e.,  $Q^1 \neq Q^2$ . This has the implication that more information about the signal parameters, namely,  $\epsilon$  and  $\psi$ , can be obtained. Equation (4.4) shows that the appropriate statistic for this network is

$$\mathbb{L} = |C_1^* Q^1 + C_2^* Q^2| \quad , \quad (4.9)$$

A search strategy may be defined by constructing templates in the space of the parameters  $\{t_c, \xi, \epsilon, \psi, \theta, \phi\}$ . Such a search will be orders of magnitude more expensive computationally

than a single detector search owing to the much larger parameter-space volume accessible here. Luckily, we can escape this plight. This is because it is possible to maximize the above quantity over  $\{\epsilon, \psi\}$  analytically. The problem of maximizing  $\mathbb{L}$  over the angles  $\{\epsilon, \psi\}$  reduces to aligning  $\mathbf{Q}$  along  $\mathbf{C}$ . It can be shown that such an alignment is always possible, regardless of the specific values one chooses for the angles  $\{\theta, \phi, \boldsymbol{\alpha}_{(1)}, \boldsymbol{\alpha}_{(2)}\}$  [1]. Consequently, the network statistic simplifies to

$$\mathbb{L}(\tau)|_{\hat{\epsilon}, \hat{\psi}} = \|\mathbf{C}(\tau)\| = (|C^1(\tau)|^2 + |C^2(\tau; \tau_{(2)})|^2)^{1/2}. \quad (4.10)$$

Note that the above statistic does not depend on  $\{\theta, \phi, \boldsymbol{\alpha}_{(1)}, \boldsymbol{\alpha}_{(2)}\}$ . Indeed, it depends only on the sum of the “ $\rho^2$ ” statistics for the two detectors evaluated at the offset times  $\tau$  and  $\tau + \tau_{(2)}$ , respectively.<sup>5</sup> Given that LLO and LHO have slightly non-identical orientations, strictly speaking the statistic derived above is the appropriate one to use for an inspiral search with an LLO-LHO network. Since the Inspiral Search Code computes the  $\rho^2$  for each detector individually, one needs an additional code that feeds these ISC outputs (with  $\tau_{(2)}$  chosen to vary in the interval  $[-10\text{msec}, 10\text{msec}]$  with discrete steps) in the above expression to evaluate the LLO-LHO network statistic. Such a code is now under development in LAL. A brief description of this code will be posted soon.

For a network of three (or more) non-coincident and non-aligned detectors, such as the one comprising of GEO, LLO, and either LHO4 or LHO2, it is once again possible to maximize  $\mathbb{L}$  in Eq. (4.4) with respect to  $\{\epsilon, \psi\}$  [1]. The resulting maximum likelihood statistic is

$$\mathbb{L}|_{\hat{\epsilon}, \hat{\psi}} = (|C^+|^2 + |C^-|^2)^{1/2}, \quad (4.11)$$

where

$$C^\pm := \mathbf{C} \cdot \hat{v}^\pm. \quad (4.12)$$

Here,  $(\hat{v}^+, \hat{v}^-)$  are two orthonormal real vectors. They form a basis in which any complex vector in  $\mathcal{H}$  can be expanded. The vectors  $\hat{v}^\pm$  depend on the detector orientations and the source-direction angles, i.e.,

$$\hat{v}^\pm = \hat{v}^\pm(\theta, \phi; \boldsymbol{\alpha}_{(1)}, \dots, \boldsymbol{\alpha}_{(M)}) , \quad (4.13)$$

(for details, refer [1,2]). Thus, the statistic (4.11) remains to be maximized over  $\{\theta, \phi\}$ . This can be done numerically by using a grid on this two-dimensional space of source-direction angles.

To use the statistic (4.11) for a coherent search using the LIGOs and GEO, one first picks a discrete set of source directions. (For details on the metric on this parameter subspace and on how it can be used to choose an optimal discretization, refer [2].) For each of these directions, one computes the basis vectors  $\hat{v}^\pm$ . This can be done using a code that is now

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<sup>5</sup>The  $\rho^2$  that the Inspiral Search Code computes by cross-correlating a template with the data of a single detector is proportional to  $|C|^2$  (where we have dropped the detector label  $I$  on  $C$ ).

available in FORTRAN. It can be transformed to LAL. Let us call this the “Helicity Basis Code” (HBC). Separately, one constructs C by collecting the cross-correlation functions  $C^{(I)}$  that are obtained for a given template by running the Inspiral Search Code through the individual data streams. For each template and source direction, one uses the computed  $C^{(I)}$  and  $\hat{v}^\pm$  in Eq. (4.11) to obtain the value of the network statistic, as a function of  $\tau$ . To compute the network statistic (4.11) over the full parameter space, one needs a code that repeats the above steps for all templates and for each (discretely sampled) source direction. Such a code takes as its inputs the outputs of the ISC (in the form of the  $C^{(I)}$ ’s) and the HBC (in the form of the  $\hat{v}^\pm$ ). This code does not yet exist in any form. Details on its algorithm will be posted soon.

## V. CONCLUSION

Given the objective of this draft, as outlined in the Introduction, we choose not to delve into other interesting issues such as probability distribution of the coherence statistic, the computational costs for a multi-detector search, etc. For a discussion on these and other related topics we refer the reader to Ref. [2].

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