The Affine Libra and the Libra of Möbius Transformations

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1 Introduction.

1.1 **General Purpose.** There exist common mathematical objects, of fundamental geometric nature, which are commonly defined through algebraic means, the algebraic structures involved being to some extent, superfluous. This paper introduces a mathematical construct which enables more direct and concise definitions of some of these objects.

1.2 **Affine Spaces.** An affine space essentially is a set, every finite subset of which has a unique "epicenter" or "center of gravity". The usual definitions involve a *vector space*, either acting on the affine space, or serving as a model of it. Vector spaces are fundamentally more complex than affine spaces—in particular they presuppose the algebraic concept of a *field*. Though inherent, a field is unnecessary in describing an affine space.

1.3 **Libras.** Libra, the Latin word for a pair of scales, is the root of the word "equilibrium". In [2] three postulates for a general concept of equilibrium were given, and a mathematical model for this concept presented, bearing the name *libra*. This mathematical model is a set L with a trinary operation $[,,]$: the object $[a,b,c]$ associated to any three elements $a,b,c \in L$ being the unique element d which must be adduced so that, when a and c are placed symmetrically on one of a pair of identical scales, and b and d are placed in corresponding positions of the other scale, then the scales are in equilibrium.

In precise mathematical terms, a **libra** consists of a set L, together with a trinary operation $[,,]$ satisfying two axioms: $[x,x,y] = y = [y,x,x]$ and $[v,w,[x,y,z]] = [[v,w,x],y,z]$ for all $v,w,x,y,z \in L$. A subset of a libra is **balanced** if it is closed relative to the operation $[,,]$. An **epicenter** of a finite sequence $\{x_i : i = 1,2,...,n\}$ in L is a unique element $e \in L$ such that $e = [x_{\sigma(1)}, e, [x_{\sigma(2)}, e, [...x_{\sigma(n)}]...]$ for each permutation σ of the first n integers. We say that a libra is **affine** if each finite sequence in the libra has an epicenter.

1.4 **Minor Theorem.** Let L be an affine space in the classical sense. For each triple $a,b,c \in L$, let $[a,b,c]$ denote the fourth vertex of the parallelogram, the first three vertices of which are a, b, and c. Then $(L,[,,])$ is an affine libra (*cf.* 4.10 *infra*).

Conversely, if $(L,[,,])$ is an affine libra, then it is also an affine space with vertices of its parallelograms being determined by the libra operation.(*cf.* 4.11 *infra*).

1.5 **Meridians.** Projective space is another example of a geometric object which is usually, but unnecessarily, defined in terms of vector spaces. It was shown in [1] that a 1-dimensional projective space (Pappian without Fano configuration) can be characterized as a set P together with a family $\mathcal{I}$ of involutions satisfying three axioms—the family $\mathcal{I}$ being just the set of involutive projective automorphisms of P. For abbreviation, we shall refer to such spaces P as *meridians*.

The family $\mathcal{J}$ of projective isomorphisms from one meridian P to another (isomorphic) meridian Q, as is well-known, is a three dimensional geometrical object. Perhaps the most familiar model of $\mathcal{J}$ is the family of linear fractional transformations $\frac{a\cdot x+b}{c\cdot x+d}$ $(a\cdot d \neq b\cdot c)$ from one field (of non-zero characteristic) onto another (*cf. Discussion* 6.1 *infra*). If, for $f, g, h \in \mathcal{J}$, one defines $[f, g, h]$ to be the composition $f \circ g^{-1} \circ h$, then $(\mathcal{J}, [,,])$ is a libra. For brevity we shall refer to such libras as *LFT libras.*

It transpires that such libras can be characterized by the behavior of points and polars relative to certain balanced subsets.

1.6 **Möbius libras.** A **coset** of a balanced set B is a set of the form $[x, B, y]$ for $x, y \in L$. The coset $[x, B, y]$ is said to be **mixed** if neither x nor y is in B. If B is not a mixed coset of itself, it is said to be **distinctive**. If a balanced set B is such that, for all distinct $x, y \in L$ there exists a coset of B which contains x but not y, then B is said to be **separating**. The **polar** of an element o of L is the set o^∞ of all elements $x \in L$ such that $x = [o, x, o]$. If L has a point o such that L has no proper balanced subset containing o^∞, and if L has a distinctive separating balanced subset B such that the intersection of o^∞ with each coset of B is balanced, and if $C \cap S \cap o^\infty$ is a singleton whenever S is a mixed coset of a coset C of B such that $C \neq [o, S, o]$, then $L, [,,]$ is said to be a **Möbius libra**.

1.7 **Major Theorem.** Necessary and sufficient for a libra to be isomorphic to an LFT libra is for it to be a Möbius libra (*cf.* 6.13 *infra*).

1.8 **3-Space as a Libra.** . Perhaps the most common Möbius libra is that in which the coefficients of the inherent linear fractional transformations are real numbers. In this case there is a natural identification of 3-dimensional real projective space S with the union of $P \times Q$ and $\mathcal{J}$. In this identification $P \times Q$ becomes an elliptic hyperboloid H, P and Q being the two reguli families of H.

The balanced subsets of $\mathcal{J}$ are the singletons of $\mathcal{J}$, the intersections of $\mathcal{J}$ with the lines of S, and the intersections of $\mathcal{J}$ with the planes of S which are tangent to H.

To obtain the polar of a point o of $\mathcal{J}$, one finds the cone C of lines through o tangent to H, and notes that the ellipse $C \cap H$ determines a plane in S: o^∞ is the intersection of this plane with $\mathcal{J}$.

The distinctive separating balanced subsets of $\mathcal{J}$ are the planes tangent to H (minus the points of tangency).

These matters, in addition to related symmetry, are discussed and illustrated in the final section of this paper.

1.9 **Program.** Section 2 introduces the basic facts about libras and their cosets, with emphasis on the contrasting *normal* and *distinctive* cosets. Section 3 exposes the groups inherent in a libra, through the medium of its left and right multiplier groups. Section 4 treats affine libras as group libras.

As a group is useful in describing a family of permutations of a set, so is a libra useful in describing a family of bijections of one set onto another. This point of view is developed in Section 5.

The geometric characterization of a one-dimensional projective space is presented in Section 6, together with the libra characterization of the family of linear fractional transformations.

Section 7 presents examples and illustrations of common meridians and libras.

1.10 **Conventions.** We shall employ the notation $:=$ for "defined as". A "generic" element is an element "arbitrarily chosen".

In the course of the paper various facts are displayed with numbers enclosed in parentheses for purposes of reference. These will be referred to as *displays.*

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2 Libras and Balanced Subsets.

2.1 **Definition.** Let L be a set and $[,,]$ a function that associates to each ordered triple (x, y, z) of elements of L a fourth element $[x, y, z]$ such that

$$[x, x, y] = y = [y, x, x] \quad (\forall x, y \in L) \tag{1}$$

and

$$[v, w, [x, y, z]] = [[v, w, x], y, z] \quad (\forall v, w, x, y, z \in L) \tag{2}$$

hold. Then L is a **set of equilibrium**, $[,,]$ is a **libra operation**, and $(L, [,,])$ is a **libra**.

2.2 **Convention.** In the present article $\boldsymbol{(L, [,,])}$ will denote a libra.

2.3 **Theorem.** *For all* $v, w, x, y, z \in L$

$$[v, [w, x, y], z] = [[v, y, x], w, z] = [v, y, [x, w, z]].$$

Proof. Applying the *Axiom Identity Equations* 1 and 2 repeatedly, we obtain

$$[v, [w, x, y], z] = [[v, y, y], [w, x, y], z] = [[v, y, [x, x, y]], [w, x, y], z] =$$

$$[[v, y, [[x, w, w], x, y]], [w, x, y], z] = [[[v, y, [x, w, w]], x, y], [w, x, y], z] =$$

$$[[[[v, y, x], w, w], x, y], [w, x, y], z] = [[[v, y, x], w, [w, x, y]], [w, x, y], z] =$$

$$[[v, y, x], w, [[w, x, y], [w, x, y], z]] = [[v, y, x], w, z] = [v, y, [x, w, z]].$$

Q.E.D

2.4 **Definition.** For $n > 1$ an integer, we define $[\boldsymbol{x_1, x_2, \ldots, x_{2n+1}}]$ inductively:

$$[x_1, x_2, \ldots, x_{2n+1}] := [[x_1, x_2, x_3, \ldots, x_{2n-1}], x_{2n}, x_{2n+1}]$$

We shall refer to such expressions $[x_1, x_2, \ldots, x_{2n+1}]$ as **libra distributions**.

2.5 **Convention.** When an entry of a distribution E is itself a distribution D of the form $[a, b, c]$, it may be useful to remove the brackets and incorporate D in E to form a distribution with two more entries than E had. In view of *Display* (2), this may be done simply by removing the brackets, *provided that D is an odd entry of E*: for example

$$[[v, w, x], y, z] = [v, w, x, y, z], \; [v, w, [x, y, z]] = [v, w, x, y, z].$$

If [a,b,c] is in an *even* entry however, it follows from *Theorem* 2.3 that the order of the elements of a, b, c must be *reversed*:

$$[v, [y, x, w], z] = [v, w, x, y, z].$$

It follows from *Display* (1) that if two adjacent entries of a distribution are identical, they can be removed from the distribution. We shall adopt such simplifications in the sequel without comment.

2.6 **Lemma.** *Let* $a, b, x, y \in L$ *satisfy* $[a, b, x] = [a, b, y]$. *Then* $x = y$.

Proof. We have

$$x = [b,a,a,b,x] = [b,a,[a,b,x]] = [b,a,[a,b,y]] = [b,a,a,b,y] = y.$$

Q.E.D

2.7 **Definition.** A subset B of L is said to be **balanced** provided $[x,y,z]$ is in B whenever $x,y,z \in B$.

2.8 **Notation.** We shall write $\boldsymbol{B \sqsubset L}$ to express that B is a balanced subset of L. For $x,y \in L$ and $S \subset L$ the expression $\boldsymbol{[x,y,S]}$ will denote the set $\{[x,y,s]: s \in S\}$. The expressions $\boldsymbol{[x,S,y]}$, and $\boldsymbol{[S,x,y]}$ are defined analogously. For $T \subset L$ we denote by $\boldsymbol{[x,S,T]}$ the set $\{[x,s,t]: s \in S, t \in T\}$. The expressions $\boldsymbol{[S,x,T]}$, and $\boldsymbol{[S,T,x]}$ are defined analogously.

2.9 **Lemma.** *Let B be a balanced subset of L, and let $b \in B$ and $c \in L$ be generic. Then*

$$[B,B,c] = [B,b,c] = [b,B,c] \quad \textit{and} \quad [c,B,B] = [c,b,B] = [c,B,b].$$

Proof. That $[B,b,c] \subset [B,B,c]$ is trivial. For generic $a,d \in B$,

$$[a,d,c] = [a,d,b,b,c] = [[a,d,b],b,c] \in [B,b,c]$$

which implies that $[B,B,c] \subset [B,b,c]$. Hence $[B,B,c] = [B,b,c]$. The remaining identities are proved similarly. *Q.E.D*

2.10 **Lemma.** *Let B be a balanced subset of L, and let $b,c \in B$ be generic. Then*

$$B = [b,c,B] = [B,c,b] = [b,B,c] = [b,B,B] = [B,B,b] = [B,B,B].$$

Proof. The inclusion $[B,B,B] \subset B$ follows from the definition of a balanced set. *Lemma* 2.10 now follows from *Lemma* 2.9. *Q.E.D*

2.11 **Proposition.** *Let $B \subset L$. The following statements are pairwise equivalent:*
(i) $B \sqsubset L$*;*
(ii) $(\forall x,y \in L)$ $[x,y,B] \sqsubset L$*;*
(iii) $(\forall x,y \in L)$ $[B,x,y] \sqsubset L$*.;*
(iv) $(\forall x,y \in L)$ $[x,B,y] \sqsubset L$*.*

Proof. [i⇒ii] Suppose (i) holds. For $a,b,c \in B$ we have

$$[[x,y,a],[x,y,b],[x,y,c]] = [x,y,a,b,y,x,x,y,c] = [x,y,a,b,y,y,c] =$$

$$[x,y,a,b,c] = [x,y,[a,b,c]]$$

which is in $[x,y,B]$ by (i). Thus (ii) holds.
[i⇒iii] Proof is analogous to that of [i⇒ii] .
[i⇒iv] Suppose that (i) holds. For $a,b,c \in B$ we have

$$[[x,a,y],[x,b,y],[x,c,y]] = [x,a,y,y,b,x,x,c,y] = [x,a,b,c,y] = [x,[c,b,a],y]$$

which is in $[x,B,y]$ by (i). Thus (iv) holds.
That each of (iii), (iv) and (iv) implies (i) follows from *Lemma* 2.10. *Q.E.D*

2.12 **Definition.** Let $B \sqsubset L$ and $x,y \in L$. The set $[x,B,y]$ is a **coset of** B, the set $[x,y,B]$ is a **left coset of** B, and the set $[B,x,y]$ is a **right coset of** B.

2.13 **Proposition.** *Let C be a coset of $B \sqsubset L$. Then any coset D of C is a coset of B.*

Proof. Choose $x, y \in L$ such that $C = [x, B, y]$. Now choose $u, v \in L$ such that $D = [u, C, v]$. For any $b \in B$ *Lemma* 2.10 implies $B = [b, B, b]$, whence

$$D = [u, [x, [b, B, b], y], v] = [u, y, b, B, b, x, v] = [[u, y, b], B, [b, x, v]].$$

Q.E.D

2.14 **Proposition.** *Let $B, C \sqsubset L$. The following statements are pairwise equivalent:*
(i) *C is a left coset of B;*
(ii) $(\forall b \in B)(\exists x \in L) \quad C = [x, b, B]$*;*
(iii) $(\forall b \in B)(\exists x \in L) \quad C = [x, B, b]$*;*
(iv) $(\exists x \in L) \quad C = [x, B, B]$.

Proof. [i⇒ii] Suppose that C is a left coset of B. Then $C = [z, y, B]$ for some $z, y \in L$. Let $b, d \in B$ be generic and let $x := [z, y, b]$. Then

$$[x, b, d] = [[z, y, b], b, d] = [z, y, d] \Longrightarrow [x, b, B] \subset C$$

and, for generic $e \in B$,

$$[z, y, e] = [z, y, b, b, e] = [x, b, e] \Longrightarrow C \subset [x, b, B],$$

which imply that (ii) holds.

[ii⇒iii] Suppose now that (ii) holds, and let x and b be as in (ii). Let $c, d \in B$ be generic. Then

$$[x, d, c] = [x, b, [b, d, c]] \Longrightarrow [x, B, c] \subset C.$$

and

$$[x, b, d] = [x, b, d, c, c] = [x, [c, d, b], c] \Longrightarrow C \subset [x, B, c],$$

which imply that (iii) holds.

[iii⇒iv] Suppose that (iii) holds, and let x and b be as in (iii). Let $c, d \in B$ be generic. Then

$$[x, c, d] = [x, c, d, b, b] = [x, [b, d, c], b] \Longrightarrow [x, B, B] \subset C.$$

That $C \subset [x, B, B]$ is trivial, which implies (iv).

[iv⇒i] Finally, suppose that (iv) holds. Let x be as in (iv) and let $c, d, y \in B$ be generic. Then

$$[x, c, d] = [x, y, [y, c, d]] \Longrightarrow C \subset [x, y, B].$$

That $[x, y, B] \subset C$ is trivial. Hence (i) holds. *Q.E.D*

2.15 **Proposition.** *Let $B, C \sqsubset L$. The following statements are pairwise equivalent:*
(i) *C is a right coset of B;*
(ii) $(\forall b \in B)(\exists x \in L) \quad C = [B, b, x]$*;*
(iii) $(\forall b \in B)(\exists x \in L) \quad C = [b, B, x]$*;*
(iv) $(\exists x \in L) \quad C = [B, B, x]$.

Proof. Repeat the proof of *Proposition* 2.14 *mutatis mutandis. Q.E.D*

2.16 **Corollary.** *Let $B \sqsubset L$. Each left coset of B and each right coset of B is a coset of B.*

Proof. Cf. Proposition 2.14 and 2.15, parts (i) and (iii). *Q.E.D*

2.17 **Definition.** A coset of $B \sqsubset L$ which is neither a left coset nor a right coset of B is a **skew coset** of B. A coset $[x, B, y]$ of B such that neither x nor y is in B is a **mixed coset**.

2.18 **Proposition.** *Let $B \sqsubset L$. The family of left cosets of B, and the family of right cosets of B, are partitions of L.*

Proof. Let $[x, y, B]$ and $[u, v, B]$ be left cosets of B. If $[x, y, c] = [u, v, d]$ for $c, d \in B$ then, for generic $b \in B$,

$$[x, y, b] = [x, y, c, c, b] = [u, v, d, c, b] = [u, v, [d, c, b]] \in [u, v, B]$$

which implies that $[x, y, B] \subset [u, v, B]$. That $[u, v, B] \subset [x, y, B]$ follows analogously. Hence follows that $[x, y, B] = [u, v, B]$.

If $z \in L$ is generic and $b \in B$, then $z = [z, b, b]$ is in the left coset $[z, b, B]$ of B. It follows that the family of left cosets is a partition of L. That the family of right cosets is as well, follows *mutatis mutandis.* Q.E.D

2.19 **Proposition.** *Let $B \sqsubset L$. The following statements are pairwise equivalent:*
(i) *B has no skew coset;*
(ii) *each left coset of B is a right coset;*
(iii) *each right coset of B is a left coset;*
(iv) $(\forall x \in L)$ *[x,B,B] = [B,B,x].*

Proof. [i⇒ii] Suppose that (i) holds, and assume there existed a left coset C of B which is not a right coset of B. If a right coset $[C, x, y]$ of C were equal to a right coset $[B, u, v]$ of B, then

$$C = [C, x, y, y, x] = [B, u, v, y, x] = [B, u, [v, y, x]]$$

which is absurd. Consequently (i) implies that the family $\mathcal{F}$ of right cosets of C is a family of left cosets of B. From *Proposition* 2.18 we know both that $\mathcal{F}$ is a partition, and since the family $\mathcal{G}$ of **all** left cosets of B is also a partition, it follows that $\mathcal{F} = \mathcal{G}$. Since B itself is in $\mathcal{G}$, it follows that B is a right coset $[C, s, t]$ of C. But then $[B, t, s] = [[C, s, t], t, s] = C$, which contradicts the assumption that C is not a right coset of B.

[i⇒iii] This follows as in the preceding paragraph, *mutatis mutandis.*

[ii⇒i] Suppose (ii) holds. Then the partition $\mathcal{G}$ of all left cosets of B is a subfamily of the partition $\mathcal{H}$ of all right cosets of B. It follows that $\mathcal{G} = \mathcal{H}$. Let $[x, B, y]$ be a generic coset of B. Then $[x, B, B]$ being a left coset of B, it equals $[B, B, a]$ for some $a \in L$. For any $b \in B$, *Proposition* 2.10 implies $B = [b, B, B]$, which yields

$$[x, B, y] = [x, B, B, b, y] = [[x, B, B], b, y] = [[B, B, a], b, y] = [B, B, [a, b, y]]$$

a right coset of B. It follows that (i) holds.

[iii⇒i] This follows as in the preceding paragraph, *mutatis mutandis.*

[iii⇒iv] Suppose that(iii) holds and that $x \in L$ is generic. From (iii) and *Proposition* 2.14 follows that, for $b \in B$, there exists $y \in L$ such that

$$[B, b, x] = [y, b, B]. \tag{3}$$

It follows that

$$B = [[B, b, x], x, b] = [[y, b, B], x, b].$$

Hence, there exists $c \in B$ such that $[y, b, c, x, b] = b$, which yields

$$[x, c, b] = [b, b, x, c, b] = [[y, b, c, x, b], b, x, c, b] = y.$$

Consequently, from *Proposition* 2.15, *Display* (3), and *Proposition* 2.10 follows

$$[B, B, x] = [B, b, x] = [y, B, B] = [x, c, b, B, B] = [x, [B, b, c], B] = [x, B, B]$$

which implies (iv).

[iv⇒iii] Trivial.

We have shown that (i) is equivalent to each of (ii) and (iii), and that (iii) is equivalent to (iv). *Q.E.D*

2.20 **Definition.** A balanced subset B of L satisfying any of the conditions of *Proposition* 2.19 is a **normal subset of** L.

2.21 **Proposition.** *If a is an element of L, then $\{a\}$ is a normal subset of L.*

Proof. That singletons are balanced is trivial. That they satisfy *Proposition* 2.19.iv follows from

$$[\{a\}, \{a\}, x] = \{[a, a, x]\} = \{x\} = \{[x, a, a]\} = [x, \{a\}, \{a\}].$$

Q.E.D

2.22 **Notation.** To denote a normal balanced subset B of L we write $\boldsymbol{B \triangleleft L}$.

2.23 **Proposition.** *Let $B \triangleleft L$ and $C \sqsubset L$ with $B \subset C$. Then $B \triangleleft C$.*

Proof. Let $x \in C$ be generic. Since B is normal in L, we have

$$[x, B, B] = [B, B, x]$$

whence follows that B is normal in C. *Q.E.D*

2.24 **Definitions.** We consider here two libras with sets of equilibrium L and M, and we abuse notation by adopting a common symbol $[,,]$ for both libra operations. A function $\phi | L \to M$ is a **libra homomorphism with domain L and codomain M** if

$$\phi([x, y, z]) = [\phi(x), \phi(y), \phi(z)] \quad (\forall x, y, z \in L).$$

Libra monomorphisms, epimorphisms, isomorphisms, and endomorphisms, respectively, are libra homomorphisms that are injective, surjective, bijective, and have identical domain and codomain, respectively. A libra endomorphism which is a libra isomorphism is a **libra automorphism**.

2.25 **Proposition.** *Let ϕ be a libra homomorphism from L to M. The following hold:*
(i) $(\forall B \sqsubset L) \quad \phi(B) \sqsubset \phi(L)$;
(ii) $(\forall B \sqsubset M) \quad \phi^{-1}(B) \sqsubset L$;
(iii) $(\forall B \triangleleft M) \quad \phi^{-1}(B) \triangleleft L$;
(iv) $(\forall m \in M) \quad \phi^{-1}(m) \triangleleft L$.

Proof. That (i) and (ii) hold is evident.

Suppose B is normal in M and let $D := \Phi^{-1}(B)$, which by (ii) is balanced. Let $[x, D, y]$ be a generic coset of D. By (i), $\phi([x, D, y]) = [\phi(x), B, \phi(y)]$ is balanced. Since B is normal in M, it follows from *Proposition* 2.23 that B is normal in $\phi(L)$, and so there exist $u, v \in L$ such that $[\phi(x), B, \phi(y)] = [\phi(u), \phi(v), B]$. We have

$$[x, D, y] = \phi^{-1}([\phi(x), B, \phi(y)]) = \phi^{-1}([\phi(u), \phi(v), B]) = [u, v, D]$$

which implies (iii).

In view of *Proposition* 2.21, (iv) follows from (iii). *Q.E.D*

2.26 **Definitions.** A **quotient space** $\mathcal{Q}$ of a libra is a maximal family of balanced sets such that, for all $A, B \in \mathcal{Q}$, A is a coset of B. For any $B \in \mathcal{Q}$ we say that $\mathcal{Q}$ **is the quotient of** L **by** B.

2.27 **Notation.** Let $\mathcal{Q}$ be a quotient space of $(L, [\,,\,,\,])$ by $B \sqsubset L$. Then we write $\mathbf{L/B}$ for $\mathcal{Q}$.

2.28 **Definition.** Let B be a normal balanced subset of L. In view of *Proposition* 2.18 and *Proposition* 2.19, we know that L/B is a partition of L and each $x \in L$ is in exactly one coset $[x, B, B]$ of B. The function $\boldsymbol{\iota_B | L \ni x \to [x, B, B] \in L/B}$ is the **quotient mapping, relative to** B**,** of L onto L/B.

2.29 **Proposition.** *Let B be a normal balanced subset of L. Let $C, D, E \in L/B$ be generic. Then*

(i) $[[c, d, e], B, B] = [[p, q, r], B, B] \quad (\forall c, p \in C,\ d, q \in D,\ e, r \in E)$.

Thus the trinary operation

(ii) $[C, D, E]_B := [[c, d, e], B, B] \quad (\forall c \in C, d \in D, e \in E)$;

is well-defined, and furthermore,

(iii) $(L/B, [\,,\,,\,]_B)$ *is a libra.*

The quotient mapping

(iv) ι_B *is an epimorphism of L onto L/B.*

Proof. By *Proposition* 2.14 there exists $x \in L$ such that $C = [x, B, B]$. Thus there exist $m, n \in B$ such that $c = [x, m, n]$. Consequently, by *Lemma* 2.10 and *Lemma* 2.9,

$$[c, B, B] = [x, m, n, B, B] = [x, m, B] = [x, B, B] = C. \tag{4}$$

Similarly, we have

$$C = [p, B, B],\ D = [d, B, B] = [q, B, B],\ E = [e, B, B] = [r, B, B]$$

so there exist $c_1, c_2, d_1, d_2, e_1, e_2 \in B$ such that $p = [c, c_1, c_2]$, $q = [d, d_1, d_2]$, and $r = [e, e_1, e_2]$. Thus, using *Lemma* 2.10 *Proposition* 2.19.iv freely,

$$[[p, q, r], B, B] = [[[c, c_1, c_2], [d, d_1, d_2], [e, e_1, e_2]], B, B] =$$

$$[c, c_1, c_2, d_2, d_1, d, e, [e_1, e_2, B], B] = [c, c_1, c_2, d_2, d_1, d, [e, B, B]] =$$

$$[c, c_1, c_2, d_2, d_1, d, [B, B, e]] = [c, c_1, c_2, d_2, d_1, [B, B, d], e] =$$

$$[c, c_1, c_2, d_2, d_1, [d, B, B], e] = [c, [c_1, [c_2, d_2, d_1], B], B, d, e] =$$

$$[c, [B, [c_2, d_2, d_1], c_1], B, d, e] = [c, B, B, d, e] = [c, [d, B, B], e] =$$

$$[c, [B, B, d], e] = [c, d, B, B, e] = [c, d, [B, B, e]] = [c, d, [e, B, B] = [[c, d, e], B, B].$$

This proves (i) and (ii).

We have

$$[C,D,D]_B = [[c,d,d],B,B] = [c,B,B] = C =$$

$$[c,B,B] = [[d,d,c],B,B] = [D,D,C]_B$$

which establishes *Display* (1) for L/B. If $F,G \in L/B$ and$f \in F$ and $g \in G$ are all generic, then

$$[[C,D,E],F,G]_B = [[[c,d,e],f,g],B,B] = [[c,d,[e,f,g]],B,B] = [C,D,[E,F,G]]_B$$

which establishes *Axiom* 2 of *Definition* 2.1 as well. Hence L/B is a libra.

That ι_B is an epimorphism of L onto L/B follows from the definition of the libra operation on L/B. *Q.E.D*

2.30 **Remarks.** The implications of the word *normal* may seem overly strong to justify use of this term. Whether this is so or not however, depends on the context. For instance, singletons are normal (*cf. Proposition* 2.21 *supra*).

Primary justification for the term is historical, and is related to the connection between libras and groups, as discussed in [2] and *Proposition* 3.6.

In the sequel there will be greater use for another type of balanced set, which may be regarded as the antithesis of a normal balanced set, and which will be defined directly. Recall that for a normal balanced subset, no cosets are skew (*cf. Proposition* 2.19.i *supra*)—in particular, no mixed coset is a skew coset.

2.31 **Proposition.** *Let B be a balanced subset of L. The following are pairwise equivalent:*
(i) *each mixed coset of B is a skew coset;*
(ii) *no non-trivial left coset of B is a right coset;*
(iii) *no non-trivial right coset of B is a left coset;*
(iv) $(\forall x,y \in L : x,y \notin B) \quad [x,B,y] \neq B$ *(i.e. B is not a mixed coset of itself);*
(v) $(\forall x,y \in L : x \notin B) \quad [x,B,y] \neq B$*;*
(vi) $(\forall x,y \in L : y \notin B) \quad [x,B,y] \neq B$*.*

Proof. [i⇒ii] Suppose that (i) holds, and let $[a,B,B]$ be a non-trivial left coset of B—that is, $a \notin B$. Assume $[a,B,B] = [B,B,c]$ for some $c \in L$ and let $b \in B$. Then

$$B = [b,B,B] = [B,B,b]$$

(*cf. Lemma* 2.10 *supra*) and so

$$B = [b,B,B,a,a] = [b,[a,B,B],a] = [b,[B,B,c],a] =$$

$$[b,c,B,B,a] = [b,c,b,b,B,B,a] = [[b,c,b],[B,B,b],a] = [[b,c,b],B,a].$$

If $[b,c,b] = d$ for some $d \in B$, then $c = [b,d,b] \in B$, so $[a,B,B] = [B,B,c] = B$, which means that a is in B: absurd. Thus [[b,c,b],B,a] is a mixed coset of B, equal to B, which violates (i). Hence (ii) holds.

[ii⇒iii] Now suppose that (ii) holds, and assume that $[B,B,c] = [a,B,B]$ for $a,c \in L$ with $c \notin B$. Evidently $a \notin B$ as well, which violates (ii). It follows that (iii) holds.

[iii⇒iv] Suppose that (iii) holds, and assume that $[x,B,y] = B$ for $x,y \in L$ with $x,y \notin B$. Let b be in B. Then $[B,B,b] = [B,b,b]$ (*cf. Lemma* 2.10 *supra*) and so

$$[B,B,[b,x,b]] = [[B,B,b],x,b] = [B,b,b,x,b] =$$

$$[y, y, B, x, b] = [y, [x, B, y], b] = [y, B, b] = [y, B, B]$$

which violates (iii). It follows that (iv) holds.

[iv⇒i] Assume there were a mixed coset $[x, B, y]$ not skew (with $x, y \notin B$). Then $[x, B, y]$ is either a right or a left coset of B. Suppose it is a right coset $[B, B, z]$. Then, for $b \in B$, we have $[B, b, z] = [B, B, z]$ (*cf. Lemma* 2.9 *supra*), whence follows $[x, B, y] = [B, b, z]$. Consequently

$$B = [[B, b, z], z, b] = [[x, B, y], z, b] = [x, B, [y, z, b]].$$

If $[y, z, b]$ were in B, then x would be in B, which is absurd. Hence B equals a mixed coset, which violates (iv). If $[B, B, z]$ were a left coset instead of a right coset, an analogous argument shows that (iv) would be violated as well.

[v⇒iv] , [vi⇒iv] Trivial.

[iv⇒v] , [iv⇒vi] Suppose that that $[x, B, y] = B$. If $y \in B$, then $[x, y, y]$is in B so$x = [[x, y, y], y, y]$ is in B. Hence, if $x \notin B$ then $[x, B, y] = B$ only if $y \notin B$ Thus (v) follows from (iv). That (vi) follows from (iv) obtains similarly.

We have shown that the four statements are pairwise equivalent. *Q.E.D*

2.32 **Definition.** A balanced subset B of L satisfying the conditions of *Propositon* 2.31 will be said to be **distinctive**.

2.33 **Proposition.** *Let $\mathcal{Q}$ be a quotient space of L. If one element of $\mathcal{Q}$ is normal, all elements are. If one element of $\mathcal{Q}$ is distinctive, all elements are.*

Proof. Suppose $B \in \mathcal{Q}$ is normal: that $[x, B, B] = [B, B, x]$ for all $x \in L$. If S is any other element of $\mathcal{Q}$, then $S = [m, B, n]$ for some $m, n \in L$—hence

$$[x, S, S] = [x, [m, B, n], [m, B, n]] = [x, n, B, m, m, B, n] = [x, n, B, B, n] = [x, [B, B, n], n] =$$

$$[x, [n, B, B], n] = [x, B, B, n, n] = [x, B, B] = [B, B, x] = [m, m, B, B, x] = [m, [B, B, m], x] =$$

$$[m, [m, B, B], x] = [m, B, B, m, x] = [m, B, n, n, B, m, x] = [[m, B, n], [m, B, n], x] = [S, S, x],$$

and consequently S is normal.

Suppose that $B \in \mathcal{Q}$ is distinctive: that $[x, B, y] \neq B$ whenever $x, y \notin B$. If S is any other element of $\mathcal{Q}$, then $S = [m, B, n]$ for some $m, n \in L$. Suppose that $[x, S, y] = S$ for $x, y \in L$. We have

$$S = [x, S, y] \Longrightarrow [m, B, n] = [x, [m, B, n], y] = [x, n, B, m, y] \Longrightarrow$$

$$[[n, x, m], B, [n, y, m]] = [n, x, [m, B, n], y, m] = [n, x, [x, n, B, m, y], y, m] =$$

$$[n, x, x, [m, B, n], y, y, m] = [n, [m, B, n], m] = [n, n, B, m, m] = B.$$

Since B is distinctive, it follows that either $[n, x, m]$ or $[n, y, m]$ is in B. Hence either $y \in [m, B, n] = S$ or $x \in [m, B, n] = S$. It follows that S is distinctive. *Q.E.D*

2.34 **Definition.** Let $\mathcal{Q}$ be a quotient space of L. If $\mathcal{Q}$ has a normal element, we say that $\mathcal{Q}$ is a **normal quotient space**. If $\mathcal{Q}$ has a distinctive element, we say that $\mathcal{Q}$ is a **distinctive quotient space**.

2.35 **Proposition.** *Let B be a distinctive balanced subset of L. Let R be non-trivial left coset of B and C a non-trivial right coset. Then there exists exactly one element of L/B such that M is a right coset of R and a left coset of C.*

Proof. Choose $l, r \in L$ and $x, y \in B$ such that $R = [l, x, B]$ and $C = [B, y, r]$. Then

$$[x, l, R] = B \implies C = [[x, l, R], y, r] = [x, l, R, y, r].$$

Letting $M := [R, y, r]$ we see that

$$[l, x, C] = [l, x, [x, l, R, y, r]] = [R, y, r] = M$$

which establishes existence.

Now suppose that $N \sqsubset L$ is another left coset of C which is also a right coset of R. Then C is a left coset of N, as well as a left coset of M, and so there exist $l_1, l_2 \in L$, $n_1 \in N$, and $m_1 \in M$ such that

$$[l_1, M, m_1] = C = [l_2, n_1, N] \implies N = [n_1, l_2, l_1, M, m_1].$$

Furthermore R is a left coset of M and a left coset of N and so there exist $r_1, r_2 \in L$, $n_2 \in N$, and $m_2 \in M$ such that

$$[m_2, M, r_1] = R = [n_2, N, r_2] \implies M = [r_1, r_2, N, n_2, m_2].$$

Substituting for N in this last equation we have

$$M = [r_1, r_2, n_1, l_2, l_1, M, m_1, n_2, m_2] = [[r_1, r_2, n_1, l_2, l_1], M, [m_1, n_2, m_2]].$$

Since M is distinctive, it follows from *Proposition* 2.31.vi that

$$[m_1, n_2, m_2] \in M \implies n_2 \in M.$$

Thus M and N have a point in common, and so, being both left cosets of C, they must be identical. *Q.E.D*

2.36 **Motivation.** Let B be a balanced subset of L, and suppose we wish to arrange the elements of L/B in an orderly way in a matrix. We place B in the upper left-hand corner. Then we compute the left cosets of B and enter them in the first column below B. Next we compute the right cosets of B and attempt to place them in the first row, to the right of B. If a right coset is not *distinct* from all the elements of the first column, then there will be repeated entries in the matrix. However, if B is *distinctive*, this will not happen. Suppose that B is distinctive and that we have constructed the first row and column. Suppose that R is an element of the first column and C an element of the first row. We wish to choose the element of L/B which lies in the same row as R and the same column as C. Recall that we obtain all the elements of L/B by choosing one element (say C) of L/B and then looking at the right cosets of the left cosets of this element. In particular, R is the right coset of a left coset M of C and, in view of *Proposition* 2.35, this M is unique. Symmetrically we could try to find the right coset of R which has a left coset equal to C. Again, by *Proposition* 2.35, this would give us the same element M. We have given a method of constructing a matrix of the elements of L/B, which is uniquely determined (once the first row and column have been chosen), *provided the balanced set B is distinctive.*

3 Multiplier Groups and Subgroups

3.1 **Definition.** The set

$$\mathcal{LM}(L) := \{\mathfrak{l}|L \to L : \ \mathfrak{l}([x,y,z]) = [\mathfrak{l}(x),y,z](\forall x,y,z \in L)\}$$

is the set of **left multipliers** of L. The set

$$\mathcal{RM}(L) := \{\mathfrak{r}|L \to L : \ \mathfrak{r}([x,y,z]) = [x,y,\mathfrak{r}(z)](\forall x,y,z \in L)\}$$

is the set of **right multipliers** of L.

3.2 **Observation.** Direct computation shows that $\mathcal{LM}$ and $\mathcal{RM}$ are groups relative to composition.

3.3 **Notation.** For $a,b \in L$ let

$$\mathfrak{l}_{a,b}(x) := [a,b,x] \quad (\forall x \in L)$$

and

$$\mathfrak{r}_{a,b}(x) := [x,a,b] \quad (\forall x \in L)).$$

3.4 **Lemma.** *For $a,b,c,d \in L$ we have*
(i) $\mathfrak{l}_{a,b} \circ \mathfrak{l}_{c,d} = \mathfrak{l}_{a,[b,c,d]} = \mathfrak{l}_{[a,b,c],d}$;
(ii) $\mathfrak{r}_{a,b} \circ \mathfrak{r}_{c,d} = \mathfrak{r}_{a,[b,c,d]} = \mathfrak{r}_{[a,b,c],d}$.

Proof. Direct computation. *Q.E.D*

3.5 **Proposition.** *The following hold:*
(i) $\mathcal{LM}(L) = \{\mathfrak{l}_{a,b} : \ a,b \in L\}$;
(ii) $\mathcal{RM}(L) = \{\mathfrak{r}_{a,b} : \ a,b \in L\}$;
(iii) $\mathcal{LM}(L) \cup \mathcal{RM}(L) \subset \{\phi : \ \phi$ *is an endomorphism of* $L\}$.

Proof. That $\mathfrak{l}_{a,b}$ is in $\mathcal{LM}$(L) follows from *Display* (2). Given $\mathfrak{l} \in L$, we have

$$\mathfrak{l}_{\mathfrak{l}(t),t}(x) = [\mathfrak{l}(t),t,x] = \mathfrak{l}([t,t,x]) = \mathfrak{l}(x) \quad (\forall t,x \ L)$$

which shows that

$$\mathfrak{l} = \mathfrak{l}_{\mathfrak{l}(t),t} \quad (\forall t \in L). \tag{5}$$

This proves (i). That (ii) holds follows *mutatis mutandis.*

To show that $\mathfrak{l} \in \mathcal{LM}(L)$ is an endomorphism, we choose $a,b \in L$ such that $\mathfrak{l} = \mathfrak{l}_{a,b}$ and observe that, for $x,y,z \in L$,

$$[\mathfrak{l}(x),\mathfrak{l}(y),\mathfrak{l}(z)] = [[a,b,x],[a,b,y],[a,b,z]] = [a,b,x,y,b,a,a,b,z] = [a,b,[x,y,z]] = \mathfrak{l}([x,y,z]).$$

That the elements of $\mathcal{RM}(L)$ are also endomorphisms is proved similarly. *Q.E.D*

3.6 **Proposition.** *Let $o \in L$ be generic and let $x \cdot y := [x,o,y]$ for all $x,y \in L$. Define*

$$\mathfrak{f}|L \ni x \to (\mathfrak{f}_x := \mathfrak{l}_{x,o}) \in \mathcal{LM}(L). \tag{6}$$

Then $(L,\cdot)$ is a group and $\mathfrak{f}$ is a group isomorphism from $(L,\cdot)$ onto $\mathcal{LM}(L)$.

Proof. For $a, b, x \in L$ we have from *Lemma* 3.4

$$\mathfrak{f}_a \circ \mathfrak{f}_b(x) = \mathfrak{l}_{a,o}(\mathfrak{l}_{b,0}(x)) = \mathfrak{l}_{[a,o,b],o}(x) = \mathfrak{f}_{a\cdot b}(x)$$

. If $a \neq o$, then $\mathfrak{f}_a(o) = [a, o, o] = a \neq o = \mathfrak{f}_o(o)$, so $\mathfrak{f}$ is injective. Furthermore,

$$\mathfrak{l}_{a,b}(x) = [a, b, x] = [[a, b, o], o, x] = \mathfrak{f}_{[a,b,o]}(x)$$

which proves $\mathfrak{f}$ is surjective. It follows that $(L, \cdot)$ is a group and $\mathfrak{f}$ is a group isomorphism. *Q.E.D*

3.7 **Proposition.** *Let $o \in L$ be generic and let $x \cdot y := [x, o, y]$ for all $x, y \in L$. Define $\mathfrak{f}|L \ni x \to \mathfrak{r}_{o,x} \in \mathcal{RM}(L)$. Then $\mathfrak{f}$ is a group anti-isomorphism from $(L, \cdot)$ onto $\mathcal{RM}(L)$.*

Proof. Analogous to the proof of *Proposition* 3.6. *Q.E.D*

3.8 **Notation.** For $B \subset L$ let $\boldsymbol{\mathcal{LM}(L; B)}$ denote the set of all $\mathfrak{l}_{a,b}$ such that $a, b \in B$, and let $\boldsymbol{\mathcal{RM}(L; B)}$ denote the set of all $\mathfrak{r}_{a,b}$ such that $a, b \in B$.

3.9 **Proposition.** *For $B \sqsubset L$, $\mathcal{LM}(L; B)$ is a subgroup of $\mathcal{LM}(L)$ and $\mathcal{RM}(L; B)$ is a subgroup of $\mathcal{RM}(L)$.*

Proof. For $a, b, c, d \in B$ *Lemma* 3.4 implies

$$\mathfrak{l}_{a,b} \circ \mathfrak{l}_{c,d} = \mathfrak{l}_{[a,b,c],d}$$

and so evidently

$$\mathfrak{l}_{a,b}^{-1} = \mathfrak{l}_{b,a}.$$

Similarly,

$$\mathfrak{r}_{a,b} \circ \mathfrak{r}_{c,d} = \mathfrak{r}_{c,[d,a,b]}$$

and

$$\mathfrak{r}_{a,b}^{-1} = \mathfrak{r}_{b,a}.$$

Q.E.D

3.10 **Notation.** Let B be a balanced subset of L. For $C, D \in L/B$ we shall write $\boldsymbol{C \overset{\mathfrak{l},B}{\sim} D}$ if $\mathcal{LM}(L; C) = \mathcal{LM}(L; D)$ and $\boldsymbol{C \overset{\mathfrak{r},B}{\sim} D}$ if $\mathcal{RM}(L; C) = \mathcal{RM}(L; D)$.

3.11 **Proposition.** *Let B be a balanced subset of L. For $C, D \in L/B$ we have*

(i) $C \overset{\mathfrak{l},B}{\sim} D \iff$ *D is a right coset of C;*

(ii) $C \overset{\mathfrak{r},B}{\sim} D \iff$ *D is a leftcoset of C.*

Proof. We prove (i). Suppose that D is a right coset of C: that $D = [C, a, b]$ for some $a, b \in L$. Then, for $d, t \in D$ there exist $c, s \in C$ such that $d = [c, a, b]$ and $t = [s, a, b]$. Thus, for $x \in L$,

$$\mathfrak{l}_{d,t}(x) = [d, t, x] = [[c, a, b], [s, a, b], x] = [c, a, b, b, a, s, x] = [c, s, x] = \mathfrak{l}_{c,s}(x).$$

This proves that $\mathcal{LM}(L; D) \subset \mathcal{LM}(L; C)$. The reverse inclusion is proved analogously. It follows that $C \overset{\mathfrak{l},B}{\sim} D$.

Now suppose that $C \overset{\mathfrak{l},B}{\sim} D$. Let $o \in D$ be fixed. For generic $a, b \in C$, select $c, d \in D$ such that $\mathfrak{l}_{a,b} = \mathfrak{l}_{c,d}$—thus

$$[a, b, o] = \mathfrak{l}_{a,b}(o) = \mathfrak{l}_{c,d}(o) = [c, d, o] \implies [C, C, o] \subset D.$$

On the other hand, if $e \in D$ is generic, we may select $s, t \in C$ such that $\mathfrak{l}_{e,o} = \mathfrak{l}_{s,t}$—thus

$$[s, t, o] = \mathfrak{l}_{s,t}(o) = \mathfrak{l}_{e,o}(o) = [e, o, o] = e \implies D \subset [C, C, o].$$

It follows that $D = [C, C, o]$. This completes the proof of (i). That (ii) holds, follows *mutatis mutandis. Q.E.D*

3.12 **Corollary.** *The relations* $\overset{\mathfrak{l},B}{\sim}$ *and* $\overset{\mathfrak{r},B}{\sim}$ *are equivalence relations.*

Proof. Follows from *Proposition* 3.11 and *Proposition* 2.18. *Q.E.D*

3.13 **Corollary.** *Let B be a balanced subset of L. The following assertions are pairwise equivalent:*
(i) *B is normal;*
(ii) *the partition induced by* $\overset{\mathfrak{l},B}{\sim}$ *is L/B;*
(iii) *the partition induced by* $\overset{\mathfrak{r},B}{\sim}$ *is L/B .*

Proof. If (i) holds, then all cosets of B are right cosets of B and of each other—hence (ii) holds by *Proposition* 3.11. Similarly, (i) implies (iii).
If (ii) holds, then *Proposition* 3.11 implies that each right coset of B is a left coset of B, whence follows that B is normal, which implies (i). That (iii) implies (i) follows similarly. *Q.E.D*

3.14 **Corollary.** *Let B be a balanced subset of L. The following assertions are pairwise equivalent:*
(i) *B is distinctive;*
(ii) *whenever either $C \overset{\mathfrak{l},B}{\sim} B$ or $C \overset{\mathfrak{r},B}{\sim} B$, then is C not a mixed coset of B.*

Proof. From *Proposition* 3.11 follows that (ii) is equivalent to *Proposition* 2.31.i. *Q.E.D*

4 Group Libras and Affine Libras

4.1 **Definition.** Let G be a group with binary operation $\cdot$. Let G act on a set S and denote the action of an element $g \in G$ on an element $s \in S$ by $g \odot s$. Suppose that

$$(7) \qquad (\forall s, t \in S)(\exists! \mathfrak{g}_{s,t} \in G) \quad \mathfrak{g}_{s,t} \odot t = s.$$

Then S is a **left group libra with action $\odot$ and action group G**. The **group libra trinary operation** $[,,]_\odot$ is defined by

$$(8) \qquad [s, t, u]_\odot := \mathfrak{g}_{s,t} \odot u \quad (\forall s, t, u \in S).$$

4.2 **Proposition.** *Let S be a left group libra with action $\odot$ and action group G. Then S is in fact a libra with libra operation $[,,]_\odot$.*

Proof. Let $s, t, u, v, w \in S$ be generic, and define $z := \mathfrak{g}_{s,t}(u)$. Then

$$\mathfrak{g}_{z,v}(v) = z = \mathfrak{g}_{s,t}(u) = \mathfrak{g}_{s,t}(\mathfrak{g}_{u,v}(v)) = (\mathfrak{g}_{s,t} \cdot \mathfrak{g}_{u,v})(v).$$

It follows from *Display* 7 that if two elements of G act the same on one element of S, they are the same. Hence $\mathfrak{g}_{z,v} = \mathfrak{g}_{s,t} \cdot \mathfrak{g}_{u,v}$, which implies

$$[[s,t,u]_\odot, v, w]_\odot = [\mathfrak{g}_{s,t}(u), v, w]_\odot = \mathfrak{g}_{z,v}(w) = (\mathfrak{g}_{s,t} \cdot \mathfrak{g}_{u,v})(w) = [s,t,[u,v,w]_\odot]_\odot.$$

Furthermore $\mathfrak{g}_{s,s}$ leaves s fixed and so, agreeing with the identity of G at s, it must be the identity. It follows that

$$[s,s,t]_\odot = \mathfrak{g}_{s,s}(t) = t = \mathfrak{g}_{t,s}(s) = [t,s,s]_\odot$$

which completes the proof that $[,,]_\odot$ is a libra operation. *Q.E.D*

4.3 **Example.** If a group G is a vector space, then a left group libra with action group G is an **affine space**.

4.4 **Proposition.** *Let G be the group $\mathcal{LM}(L)$ of left multipliers of a generic libra L. For $\mathfrak{l} \in \mathcal{LM}(L)$ and $x \in L$, define*

$$\mathfrak{l} \odot x := \mathfrak{l}(x). \tag{9}$$

Then L is a left group libra with action group $\mathcal{LM}(L)$.

Proof. For $x, y \in L$, define $\mathfrak{g}_{x,y} := \mathfrak{l}_{x,y}$ (*cf. Notation* 3.3). Then

$$\mathfrak{g}_{x,y} \odot y = \mathfrak{g}_{x,y}(y) = [x,y,y] = x$$

and, if $\mathfrak{l} \odot y = x$ for some $\mathfrak{l} \in \mathcal{LM}(L)$ as well, then $\mathfrak{l} = \mathfrak{l}_{x,y}$ by *Display* (5). *Q.E.D*

4.5 **Proposition.** *Let S be a left group libra with action group G. For each $g \in G$ define $\gamma(g)|S \ni s \to g \odot s \in S$. Then γ is a group isomorphism of G onto $\mathcal{LM}(S)$.*

Proof. Let $m \in G$ be generic. For $s, t, u \in S$,

$$(m \cdot \mathfrak{g}_{s,t}) \odot t = m \odot (\mathfrak{g}_{s,t} \odot t) = m \odot s = \mathfrak{g}_{m \odot s, t} \odot t$$

which implies that $m \cdot \mathfrak{g}_{s,t} = \mathfrak{g}_{m \odot s, t}$. This, with *Display* 8, yields

$$m \odot [s,t,u]_\odot = m \odot (\mathfrak{g}_{s,t} \odot u) = (m \cdot \mathfrak{g}_{s,t}) \odot u = \mathfrak{g}_{m \odot s, t} \odot u = [m \odot s, t, u]_\odot$$

which implies that $\gamma(m)$ is a left multiplier of S: that γ maps G into $\mathcal{LM}(S)$.

That γ is a group homomorphism is evident so, to complete the proof, it will suffice to show that γ is injective and surjective. If $m, n \in G$ are distinct, then *Display* 7 implies that $\gamma(m)$ and $\gamma(n)$ agree on no element of S—hence γ is injective.

Let $\mathfrak{m} \in \mathcal{LM}(S)$ be generic, and let $s \in S$. Then *Display* 5 implies that $\mathfrak{m} = \mathfrak{l}_{\mathfrak{m}(s),s}$. Since

$$\mathfrak{m}(s) = \mathfrak{l}_{\mathfrak{m}(s),s}(s) = \mathfrak{g}_{\mathfrak{m}(s),s} \odot s = (\gamma(\mathfrak{g}_{\mathfrak{m}(s),s}))(s),$$

it follows that $\mathfrak{m} = \gamma(\mathfrak{g}_{\mathfrak{m}(s),s})$, and γ is surjective. *Q.E.D*

4.6 **Discussion.** From *Example* 4.3 and *Proposition* 4.4 follows that affine spaces are precisely the libras of which the groups of left multipliers are vector spaces over some field. It happens however that there is a more direct means of determining whether any given libra is an affine space, and for this we require the idea of an "epicenter".

4.7 **Definition.** Let $\{x_1, x_2, ..., x_n\}$ be the range of a finite sequence of elements of L. If there exists a unique $e \in L$ such that

$$[x_{\sigma(1)}, e, x_{\sigma(2)}, e, ..., x_{\sigma(n)}] = e \tag{10}$$

whenever σ is a permutation of the first n integers, then e is an **epicenter of the sequence** x.

If each non-void finite sequence of elements of L has an epicenter, then L by definition is an **affine libra**.

4.8 **Lemma.** *Let $o \in L$ be generic and let $\cdot$ be the group operation on L as in* Proposition *3.6:*

$$x \cdot y := [x, o, y] \qquad (\forall x, y \in L). \tag{11}$$

Then, relative to this operation we have, for any odd integer n and generic $x_1, x_2, ..., x_n \in L$,

$$[x_1, x_2, ..., x_n] = x_1 \cdot x_2^{-1} \cdot ... \cdot x_{n-1}^{-1} \cdot x_n$$

(where every other term on the right side of the equation is an inverse).

Proof. We shall prove by induction on n. For $n = 1$ the statement is trivial. Suppose the statement is known to be true for a given $n-2$, n odd. Then

$$[x_1, x_2, ..., x_n] = [[x_1, x_2, ..., x_{n-1}, o], o, x_n] = (x_1 \cdot x_2^{-1} \cdot ... \cdot x_{n-1}^{-1} \cdot o) \cdot x_n = x_1 \cdot x_2^{-1} \cdot ... \cdot x_{n-1}^{-1} \cdot x_n$$

since o is the identity in this context. *Q.E.D*

4.9 **Lemma.** *Let $o \in L$ be generic, and let $\{x_1, x_2, ..., x_n\}$ be a generic sequence in L of odd cardinality. Then*

$$\mathfrak{l}_{[x_1, x_2, ..., x_n], o} = \mathfrak{l}_{x_1, o} \circ \mathfrak{l}_{x_2, o}^{-1} \circ ... \circ \mathfrak{l}_{x_n, o}$$

(where every other term on the right side of the equation is an inverse function).

Proof. By *Proposition* 3.6 the mapping $\mathfrak{f} | L \ni x \to \mathfrak{l}_{x,o} \in \mathcal{LM}(L)$ is a group isomorphism when L has the group operation defined in *Display* 11. *Lemma* 4.9 now follows from *Lemma* 4.8. *Q.E.D*

4.10 **Proposition.** *Let A be an affine space (i.e. A is a left group libra with acting group V a vector space over some field Φ). Then A is an affine libra.*

Furthermore epicenters in the libra sense are the same as epicenters in the usual sense of an affine space.

Proof. In view of *Propositions* 4.4 and 4.5 we may (and shall) take $\mathcal{LM}(A)$ for V. Since V is commutative, we shall write its group operation as $+$. Let $o, e \in A$. Let $\{x_1, x_2, ..., x_n\}$ be a finite sequence in A. In view of *Proposition* 3.6,

$$[x_1, e, x_2, e, ..., x_n] = e \iff \mathfrak{l}_{[x_1, e, x_2, e, ..., x_n], o} = \mathfrak{l}_{e,o}.$$

By *Lemma* 4.9

$$\mathfrak{l}_{[x_1, e, x_2, e, ..., x_n], o} = \mathfrak{l}_{x_1, o} - \mathfrak{l}_{e,o} + \mathfrak{l}_{x_2, o} - \mathfrak{l}_{e,o} + ... + \mathfrak{l}_{x_n, o}.$$

It follows that

$$[x_1, e, x_2, e, ..., x_n] = e \iff \mathfrak{l}_{x_1, o} - \mathfrak{l}_{e,o} + \mathfrak{l}_{x_2, o} - \mathfrak{l}_{e,o} + ... + \mathfrak{l}_{x_n, o} = \mathfrak{l}_{e,o}$$

$$\iff \frac{\mathfrak{l}_{x_1, o} + \mathfrak{l}_{x_2, o} + ... + \mathfrak{l}_{x_n, o}}{n} = \mathfrak{l}_{e,o}.$$

Q.E.D

4.11 **Theorem.** *A libra L is an affine libra if and only if $\mathcal{LM}(L)$ is a vector space (i.e. L is an affine space).*

Proof. In view of *Proposition* 4.10 we need only show that an affine libra is an affine space. Suppose then that L is an affine libra. For a finite sequence $x_1, x_2, ..., x_n$ we shall write $\epsilon\pi\iota(x_1, x_2, ..., x_n)$ for its epicenter.

Let $o \in L$ and let $\mathfrak{f}$ be as in *Display* 6. Here we find it convenient to write the value of $\mathfrak{f}$ at $x \in L$ as $\mathfrak{f}(x)$ rather than as $\mathfrak{f}_x$. *Proposition* 3.6 states that $\mathfrak{f}$ is an isomorphism. We first show that $\mathcal{LM}(L)$ is commutative. Let $x, y \in L$ be generic and consider the sequence with range $(x, [o, x, o], y, [o, y, o])$. The equation

$$[x, o, [o, x, o], o, y, o, [o, y, o]] = [x, o, o, x, o, o, y, o, o, y, o] = o$$

shows that $o = \epsilon\pi\iota(x, [o, x, o], y, [o, y, o])$. Consequently $o = \epsilon\pi\iota(y, [o, x, o], [o, y, o], x)$ as well:

$$o = [y, o, [o, x, o], o, [o, y, o], o, x] = [y, o, o, x, o, o, o, y, o, o, x] = [y, x, o, y, x]$$

which implies

$$[x, o, y] = [x, [y, x, o, y, x], y] = [x, x, y, o, x, y, y] = [y, o, x].$$

Hence

$$\mathfrak{f}(x) \circ \mathfrak{f}(y) = \mathfrak{l}_{x,o} \circ \mathfrak{l}_{y,o} = \mathfrak{l}_{y,o} \circ \mathfrak{l}_{x,o} = \mathfrak{f}(y) \circ \mathfrak{f}(x).$$

It follows that $\mathcal{LM}(L)$ is a commutative group—consequently we shall write $+$ for its binary operation.

Let $\mathbf{E}$ denote the ring of endomorphisms of $\mathcal{LM}(L)$—we denote by $+$ and $\cdot$ its binary additive and multiplicative operations. For a positive integer n, we define

$$\mathbf{n}|\mathcal{LM}(L) \ni \mathfrak{x} \to \overbrace{\mathfrak{x} + \mathfrak{x} + ... + \mathfrak{x}}^{n} \in \mathcal{LM}(L)$$

which is evidently in $\mathbf{E}$. For a positive integer n, we define

$$\tfrac{\mathbf{1}}{\mathbf{n}}|\mathcal{LM}(L) \ni \mathfrak{x} \to \mathfrak{f}(\epsilon\pi\iota(\overbrace{o, o, ..., o}^{n-1}, \mathfrak{f}^{-1}(\mathfrak{x}))) \in \mathcal{LM}(L).$$

For $x \in L$ we have

$$x = [\overbrace{o, x, o, x, o, ..., o}^{2n-3}, x, [\overbrace{x, o, x, o, x, ..., x}^{2n-1}]]$$

which implies that

$$x = \epsilon\pi\iota(\overbrace{o, o, ..., o}^{n-1}, [\overbrace{x, o, x, o, x, ..., x}^{2n-1}]),$$

whence follows that

$$\mathfrak{f}(x) = \frac{\mathbf{1}}{\mathbf{n}}(\mathbf{n}(\mathfrak{f}(x))). \tag{12}$$

In particular, this implies that $\frac{\mathbf{1}}{\mathbf{n}}$ is surjective. If $\mathfrak{f}(e) = \frac{\mathbf{1}}{\mathbf{n}}(\mathfrak{f}(x)) = \frac{\mathbf{1}}{\mathbf{n}}(\mathfrak{f}(y))$ for $x, y \in L$, then

$$[\overbrace{o, e, o, e, o, ..., o}^{2n-3}, e, x] = [\overbrace{o, e, o, e, o, ..., o}^{2n-3}, e, y]$$

whence, by *Lemma* 2.6, follows that $x = y$. Thus $\frac{\mathbf{1}}{\mathbf{n}}$ is bijective, and so an isomorphism. From *Display* 12 it follows that $\frac{\mathbf{1}}{\mathbf{n}}$ is the inverse of $\mathbf{n}$.

Let Φ be the subring of $\mathbf{E}$ generated by all the isomorphisms $\mathbf{n}$ and $\frac{\mathbf{1}}{\mathbf{n}}$. Standard manipulation with fractions shows that the general element of Φ is of the form $\frac{\mathbf{1}}{\mathbf{n}} \circ \mathbf{m}$ for integers m and n. Consequently Φ is a field either isomorphic to the field of rational numbers or to a field of prime order. Hence $\mathcal{LM}$(L) is a vector space over Φ. *Q.E.D*

5 Correspondence Libras

5.1 **Definition.** Let X and Y be sets and F a non-void family of bijections from X onto Y such that $x \circ y^{-1} \circ z$ is in F whenever $x, y, z \in F$. We say that F is a **correspondence set of equilibrium** with **domain** X and **co-domain** Y. The **correspondence libra operation** is

$$[x, y, z] := x \circ y^{-1} \circ z \qquad (\forall x, y, z \in F).$$

The **correspondence libra** is the ordered pair $(F, [\,,\,,\,])$.

Let n be a positive integer, and suppose whenever $\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_\mathsf{n} \in \mathsf{X}$ and $\mathsf{y}_1, \mathsf{y}_2, \ldots, \mathsf{y}_\mathsf{n} \in \mathsf{Y}$, there exists $x \in F$ such that $x(\mathsf{x}_\mathsf{i}) = \mathsf{y}_\mathsf{i}$ for each $\mathsf{i} = 1, 2, \ldots, \mathsf{n}$. Then F and the correspondence libra will be said to be **homogeneous of degree n**. If n is 1, we say that F is **homogeneous**.

5.2 **Notation.** Let F be a correspondence set of equilibrium with domain X and co-domain Y. We define

$$\mathfrak{LM}(F) := \{x \circ y^{-1} : \ x, y \in F\} \text{ and } \mathfrak{RM}(F) := \{x^{-1} \circ y : \ x, y \in F\}.$$

5.3 **Observation.** The sets $\mathfrak{LM}(F)$ and $\mathfrak{RM}(F)$ are groups relative to the binary operation of composition.

5.4 **Proposition.** *Let F be a correspondence set of equilibrium with domain X and co-domain Y. Then*

(i) *the mapping $\lambda|\mathcal{LM}(F) \ni \mathfrak{l}_{x,y} \to x \circ y^{-1} \in \mathfrak{LM}(F)$ is a well-defined group isomorphism;*

(ii) *the mapping $\rho|\mathcal{RM}(F) \ni \mathfrak{r}_{x,y} \to x^{-1} \circ y \in \mathfrak{RM}(F)$ is a well-defined group isomorphism.*

Proof. We shall demonstrate only (i), the proof to (ii) being entirely analogous. Let $\mathfrak{l}_{x,y} = \mathfrak{l}_{r,s}$ for $x, y, r, s \in F$. Then, for each $t \in F$,

$$x \circ y^{-1} \circ t = [x, y, t] = \mathfrak{l}_{x,y}(t) = \mathfrak{l}_{r,s}(t) = [r, s, t] = r \circ s^{-1} \circ t$$

which implies that $x \circ y^{-1} = r \circ s^{-1}$. Thus λ is well-defined.

For $w, x, y, z \in F$ we have, using *Lemma* 3.4

$$\lambda(\mathfrak{l}_{w,x} \cdot \mathfrak{l}_{y,z}) = \lambda(\mathfrak{l}_{[w,x,y],z}) = [w, x, y] \circ z^{-1} = w \circ x^{-1} \circ y \circ z^{-1} = \lambda(\mathfrak{l}_{w,x}) \circ \lambda(\mathfrak{l}_{y,z})$$

which shows that λ is an epimorphism.

If $\mathfrak{l}_{w,x} \neq \mathfrak{l}_{y,z}$, then there exists $a \in L$ with $[w, x, a] \neq [y, z, a]$, and so $w \circ x^{-1} \neq y \circ z^{-1}$. Hence λ is an isomorphism. *Q.E.D*

5.5 **Notation.** Let F be a correspondence set of equilibrium with domain X and co-domain Y. Let $\mathsf{x} \in \mathsf{X}$ and $\mathsf{y} \in \mathsf{Y}$ be generic. We write $\boldsymbol{F}_{\mathsf{y},\mathsf{x}}$ for the set of all $x \in F$ such that $x(\mathsf{x}) = \mathsf{y}$.

5.6 **Proposition.** *Let F be a correspondence set of equilibrium with domain X and co-domain Y. Let $\mathsf{x} \in \mathsf{X}$ and $\mathsf{y} \in \mathsf{Y}$ be generic. Then*

(i) *$F_{\mathsf{y},\mathsf{x}}$ is balanced;*

(ii) $(\forall a, b \in F) \qquad [a, F_{\mathsf{y},\mathsf{x}}, b] = F_{a(\mathsf{x}), b^{-1}(\mathsf{y})}$.

Proof. That (i) holds follows directly from the definitions.

For $a, b \in F$ and $f \in F_{\mathsf{y},\mathsf{x}}$ we have

$$[a, f, b](b^{-1}(\mathsf{y})) = a(f^{-1}(b(b^{-1}(\mathsf{y})))) = a(f^{-1}(\mathsf{y})) = a(\mathsf{x})$$

which proves $[a, F_{\mathsf{x},\mathsf{y}}, b] \subset F_{a(\mathsf{x}), b^{-1}(\mathsf{y})}$.

For $a, b \in F$ and $h \in F_{a(\mathsf{x}), b^{-1}(\mathsf{y})}$ let $g := [b, h, a]$. Then

$$g(\mathsf{x}) = b(h^{-1}(a(\mathsf{x}))) = b(b^{-1}(\mathsf{y})) \implies g \in F_{\mathsf{y},\mathsf{x}}.$$

Since $[a, g, b] = a \circ (b \circ h^{-1} \circ a)^{-1} \circ b = h$, it follows that $F_{a(\mathsf{x}), b^{-1}(\mathsf{y})} \subset [a, F_{\mathsf{x},\mathsf{y}}, b]$. Thus (ii) holds. *Q.E.D*

5.7 **Notation.** Let F be a correspondence set of equilibrium with domain X and co-domain Y. We write $\boldsymbol{\mathcal{Q}(F)}$ for the family of all $F_{\mathsf{y},\mathsf{x}}$ with $\mathsf{x} \in \mathsf{X}$ and $\mathsf{y} \in \mathsf{Y}$.

5.8 **Proposition.** *Let F be a correspondence set of equilibrium with domain X and co-domain Y. Let $\mathsf{x} \in \mathsf{X}, \mathsf{y} \in \mathsf{Y}$ be generic. Then*
(i) $\mathcal{Q}(F) = F/F_{\mathsf{y},\mathsf{x}}$ *if F is homogeneous;*
(ii) $F_{\mathsf{y},\mathsf{x}}$ *is distinctive if F is homogeneous of degree 2 and X has more than two points.*

Proof. Suppose that F is homogeneous, and let $\mathsf{r} \in \mathsf{X}, \mathsf{s} \in \mathsf{Y}$ be generic. Choose $x, y \in F$ such that $x(\mathsf{x}) = \mathsf{s}$ and $y(\mathsf{r}) = \mathsf{y}$. Then *Proposition* 5.6.(ii) implies that $[x, F_{\mathsf{y},\mathsf{x}}, y] = F_{\mathsf{s},\mathsf{r}}$. Hence $\mathcal{Q}(F)$ is the set of cosets of $F_{\mathsf{x},\mathsf{y}}$, which proves (i).

Now suppose that F is homogeneous of degree 2 and X has more than two points. Assume that $\mathcal{Q}(F)$ were not distinctive: that there existed a left coset G of $F_{\mathsf{y},\mathsf{x}}$ and a non-trivial right coset H of $F_{\mathsf{y},\mathsf{x}}$ such that $G = H$. Then for some $x, y \in F$,

$$[x, F_{\mathsf{y},\mathsf{x}}, F_{\mathsf{y},\mathsf{x}}] = G, \qquad [F_{\mathsf{y},\mathsf{x}}, F_{\mathsf{y},\mathsf{x}}, y] = H. \tag{13}$$

Thus, for any $f \in F_{\mathsf{y},\mathsf{x}}$, from *Lemma* 2.10

$$F_{\mathsf{y},\mathsf{x}} = [y, y, F_{\mathsf{y},\mathsf{x}}, F_{\mathsf{y},\mathsf{x}}, f] = [y, H, f] = [y, G, f] = [y, F_{\mathsf{y},\mathsf{x}}, F_{\mathsf{y},\mathsf{x}}, x, f]. \tag{14}$$

Since H is a non-trivial right coset, G is a non-trivial coset as well—it follows from *Display* 13 that neither x nor y is in $F_{\mathsf{y},\mathsf{x}}$.

Since X and Y have the same cardinality, Y has more than two points. Consequently there exists $\mathsf{z} \in \mathsf{Y}$ such that $\mathsf{z} \neq \mathsf{y}$ and $\mathsf{z} \neq y(\mathsf{x})$. Let $d \in F_{\mathsf{y},\mathsf{x}}$ be generic. Then

$$d(y^{-1}(\mathsf{z})) \neq d(y^{-1}(y(\mathsf{x})) = d(\mathsf{x}) = \mathsf{y}.$$

Since x is not in $F_{\mathsf{y},\mathsf{x}}$, we have $x^{-1}(\mathsf{y}) \neq \mathsf{x}$ and so, since F is homogeneous of degree 2, there exists $e \in F_{\mathsf{y},\mathsf{x}}$ such that $e(x^{-1}(\mathsf{y})) = d(y^{-1}(\mathsf{z}))$. Let $f \in F_{\mathsf{y},\mathsf{x}}$ be generic. We have

$$[y, d, e, x, f](\mathsf{x}) = y(d^{-1}(e(x^{-1}(f(\mathsf{x}))))) = y(d^{-1}(e(x^{-1}(\mathsf{y})))) = y(d^{-1}(d(y^{-1}(\mathsf{z})))) = \mathsf{z}.$$

¿From *Display* (14) we have $[y, d, e, x, f] \in F_{\mathsf{y},\mathsf{x}}$, whence follows

$$[y, d, e, x, f](\mathsf{x}) = \mathsf{y} \neq \mathsf{z}$$

which is absurd. Consequently $G \neq H$, which proves (ii). *Q.E.D*

5.9 **Observation.** We seek for a criterion which will insure that an abstract libra is isomorphic to some correspondence libra F with quotient space $\mathcal{Q}(F)$. In view of *Proposition* 5.8, we consider libras with distinctive balanced sets.

5.10 **Definition.** Let $\mathcal{Q}$ be a quotient space of L. An **abstract point in the context of** $\mathcal{Q}$ is a partition of L consisting of elements of $\mathcal{Q}$. If an abstract point consists of the left cosets of some $B \in \mathcal{Q}$, we say that it is an **abstract ordinate point**. If an abstract point consists of the right cosets of some $B \in \mathcal{Q}$, we say that it is an **abstract co-ordinate point**. The collection of ordinate points is the **abstract domain in the context of** $\mathcal{Q}$, and the collection of co-ordinate points is the **abstract co-domain in the context of** $\mathcal{Q}$.

5.11 **Notation.** Let $\mathcal{Q}$ be a quotient space of L. The abstract domain in the context of $\mathcal{Q}$ is denoted $\boldsymbol{\mathsf{X}_{\mathcal{Q}}}$ and the abstract co-domain in the context of $\mathcal{Q}$ is denoted $\boldsymbol{\mathsf{Y}_{\mathcal{Q}}}$.

For $B \in \mathcal{Q}$ we write $\underline{\boldsymbol{B}}$ for the element of $\mathsf{X}_{\mathcal{Q}}$ containing B, and $\overline{\boldsymbol{B}}$ for the element of $\mathsf{Y}_{\mathcal{Q}}$ containing B.

5.12 **Proposition.** *Let $\mathcal{Q}$ be a quotient space of L. Let x be a generic abstract ordinate point and y be a generic abstract co-ordinate point. Then*
(i) $(\forall \mathsf{x} \in \mathsf{X}_{\mathcal{Q}}, \mathsf{y} \in \mathsf{Y}_{\mathcal{Q}})\ \ \mathsf{x} \cap \mathsf{y} \neq \phi$*;*
(ii) $\mathcal{Q}$ *is distinctive* $\iff$ $(\forall \mathsf{x} \in \mathsf{X}_{\mathcal{Q}}, \mathsf{y} \in \mathsf{Y}_{\mathcal{Q}})\ \ \mathsf{x} \cap \mathsf{y}$ *is a singleton.*

Proof. Let $\mathsf{x} \in \mathsf{X}_{\mathcal{Q}}$ and $\mathsf{y} \in \mathsf{Y}_{\mathcal{Q}}$. Let $A \in \mathsf{x}$ and $B \in \mathsf{y}$ be generic. Since B is a coset of A, there exist $u, v \in L$ $B = [u, A, v]$. For $b \in B$ we have

$$[B, b, u] = [b, B, u] = [b, [u, A, v], u] = [b, v, A, u, u] = [b, v, A]$$

which implies that $[B, b, u]\mathsf{y} \cap \mathsf{x}$. This proves (i). Now suppose that $\mathcal{Q}$ is distinctive. Let $\mathsf{x} \in \mathsf{X}_{\mathcal{Q}}$ and $\mathsf{y} \in \mathsf{Y}_{\mathcal{Q}}$. Let $S, T \in \mathsf{x} \cap \mathsf{y}$ be generic. Then there exist $m, n \in L$ such that $T = [m, S, n]$. From *Proposition* 2.31.v and vi it follows that m and n are in S. But this means that $S = T$, which implies that $\mathsf{x} \cap \mathsf{y}$ is a singleton. Finally, suppose that $\mathcal{Q}$ iw not distinctive. By *Proposition* 2.31.iv there exists $S \in \mathcal{Q}$ and $x, y \notin S$ such that $S = [x, S, y]$. For $s \in S$ we have

$$[y, s, S] = [y, S, s] = [y, [x, S, y], s] = [y, y, S, x, s] = [S, x, s].$$

But this means than $\{S, [y, s, S]\} \in \underline{S} \cap \overline{S}$. This completes the proof of (ii). *Q.E.D*

5.13 **Observation.** From *Proposition* 5.12.ii follows that a distinctive quotient space is in bijective correspondence with the Cartesian product of the the abstract domain and the abstract codomain.

5.14 **Notation.** For each $x \in L$, we define $\breve{\boldsymbol{x}}^{\boldsymbol{\mathcal{Q}}}|\boldsymbol{\mathcal{Q}} \ni \boldsymbol{B} \to [\boldsymbol{x}, \boldsymbol{B}, \boldsymbol{x}] \in \boldsymbol{\mathcal{Q}}$, $\acute{\boldsymbol{x}}^{\boldsymbol{\mathcal{Q}}}|\boldsymbol{\mathcal{Q}} \ni \boldsymbol{B} \to [\boldsymbol{x}, \boldsymbol{B}, \boldsymbol{B}] \in \boldsymbol{\mathcal{Q}}$, and $\grave{\boldsymbol{x}}^{\boldsymbol{\mathcal{Q}}}|\boldsymbol{\mathcal{Q}} \ni \boldsymbol{B} \to [\boldsymbol{B}, \boldsymbol{B}, \boldsymbol{x}] \in \boldsymbol{\mathcal{Q}}$.

5.15 **Lemma.** *Let $B \sqsubset L$ and $x, a \in L$ be generic. Then*
(i) *[x,[a,B,B],x] = [[x,B,B],[x,B,B],[x,a,x]];*
(ii) *[x,[B,B,a],x] = [[x,a,x],[B,B,x],[B,B,x]];*
(iii) *[x,[a,B,B],[a,B,B]]=[x,B,B];*
(iv) *[[B,B,a],[B,B,a],x]=[B,B,x].*

Proof. We have

$$[x, [a, B, B], x] = [x, B, B, a, x] =$$

$$[x, B, B, B, B, x, x, a, x] = [[x, B, B], [x, B, B], [x, a, x]]$$

and

$$[x, [B, B, a], x] = [x, a, B, B, x] =$$

$$[x, a, x, x, B, B, B, B, x] = [[x, a, x], [B, B, x], [B, B, x]]$$

which establish (i) and (ii). Direct simplification yields (iii) and (iv). *Q.E.D*

5.16 **Proposition.** *Let $\mathcal{Q}$ be a quotient space of L. Then, for $x \in L$,*
(i) $(\forall \mathsf{x} \in \mathsf{X}_{\mathcal{Q}})(\forall B \in \mathsf{x})$ $\{\breve{x}^{\mathcal{Q}}(S) : S \in \mathsf{x}\} = \overline{[x, B, B]}$*;*
(ii) $(\forall \mathsf{y} \in \mathsf{Y}_{\mathcal{Q}})(\forall B \in \mathsf{y})$ $\{\breve{x}^{\mathcal{Q}}(S) : S \in \mathsf{y}\} = \underline{[B, B, x]}$*.;*
(iii) $(\forall \mathsf{x} \in \mathsf{X}_{\mathcal{Q}})(\forall B, S \in \mathsf{x})$ $\acute{x}^{\mathcal{Q}}(S) = \acute{x}^{\mathcal{Q}}(B)$*;*
(iv) $(\forall \mathsf{y} \in \mathsf{Y}_{\mathcal{Q}})(\forall B, S \in \mathsf{y})$ $\grave{x}^{\mathcal{Q}}(S) = \grave{x}^{\mathcal{Q}}(B)$*.*

Proof. *Lemma* 5.15.i implies (i) and *lemma* 5.15.ii implies (ii). *Lemma* 5.15.iii implies (iii) and *Lemma* 5.15.iv implies (iv). *Q.E.D*

5.17 **Notation.** Let $\mathcal{Q}$ be a quotient space of L and let $x \in L$. In view of *Proposition* 5.16.iii and *Proposition* 5.16.iv we may define

$$\dot{x}|\mathsf{X}_{\mathcal{Q}} \to \mathcal{Q} \text{ by } \dot{x}(\underline{B}) := [x, B, B] \quad (\forall B \in \mathcal{Q})$$

and

$$\ddot{x}|\mathsf{Y}_{\mathcal{Q}} \to \mathcal{Q} \text{ by } \ddot{x}(\overline{B}) := [B, B, x] \quad (\forall B \in \mathcal{Q}).$$

Using these, we define

$$\overline{x}^{\mathcal{Q}}|\mathsf{X}_{\mathcal{Q}} \ni \mathsf{x} \to \overline{\dot{x}(\mathsf{x})} \in \mathsf{Y}_{\mathcal{Q}}$$

and

$$\underline{x}_{\mathcal{Q}}|\mathsf{Y}_{\mathcal{Q}} \ni \mathsf{y} \to \underline{\ddot{x}(\mathsf{y})} \in \mathsf{X}_{\mathcal{Q}}.$$

We write $\boldsymbol{L_{\mathcal{Q}}}$ for the set $\{\overline{x}^{\mathcal{Q}} : x \in L\}$.

5.18 **Lemma.** *For $x \in L$ we have*

$$\overline{x}^{\mathcal{Q}} = (\underline{x}_{\mathcal{Q}})^{-1}.$$

Proof. For $\mathsf{x} \in \mathsf{X}_{\mathcal{Q}}$ we have, for $B \in \mathsf{x}$

$$\underline{x}_{\mathcal{Q}}(\overline{x}^{\mathcal{Q}}(\mathsf{x})) = \underline{x}_{\mathcal{Q}}(\overline{\dot{x}(\mathsf{x})}) = \underline{x}_{\mathcal{Q}}(\overline{[x, B, B]}) = \underline{\ddot{x}(\overline{[x, B, B]})} =$$

$$\underline{[[x, B, B], [x, B, B], x]} = \underline{[x, B, B, B, B, x, x]} = \underline{[x, B, B]} = \mathsf{x}.$$

That $\overline{x}^{\mathcal{Q}}(\underline{x}_{\mathcal{Q}}(\mathsf{y})) = \mathsf{y}$ for generic $\mathsf{y} \in \mathsf{Y}_{\mathcal{Q}}$ follows analogously. *Q.E.D*

5.19 **Interpretation and Symmetry.** Suppose that $\mathcal{Q}$ is a distinctive quotient space, and construct a matrix $\mathcal{M}$ of the elements of $\mathcal{Q}$ as in *Motivation* 2.36. Let o be in L. The function $\dot{o}$ is a "choice function": choosing one element from each column $\mathcal{C}$ of $\mathcal{M}$. The function $\overline{o}^{\mathcal{Q}}$ takes a column $\mathcal{C}$, chooses an element of that column, and then selects the row $\mathcal{R}$ of that chosen element. The function $\breve{o}^{\mathcal{Q}}$ takes the individual elements of the column $\mathcal{C}$ and maps them to the elements of the row $\mathcal{R}$.

The matrix constructed in 2.36 was completely determined by selecting the upper-left element $M_{1,1}$ and enumerating the elements of its column $\overline{M_{1,1}}$ and its row $\underline{M_{1,1}}$ (beginning with $M_{1,1}$ in each enumeration). The freedom to determine in advance $M_{1,1}$ and the rest of the order of the elements of the first row may be exchanged for the security of having the matrix symmetric relative to the mapping $\breve{o}^{\mathcal{Q}}$ as follows:

First an abstract ordinate point p is selected to be be first column. Since $\breve{o}^{\mathcal{Q}}(\mathsf{p}) = [o, \mathsf{p}, o]$ is a abstract coordinate point (or row), it has one point in common with p—this will be $M_{1,1}$. Now p is given an enumeration $\{M_{1,1}, M_{2,1}, ...\}$. Symmetry relative to $\breve{o}^{\mathcal{Q}}$ requires the first row to be $\{M_{1,1} = [o, M_{1,1}, o], [o, M_{2,1}, o], ...\}$. As in 2.36, for indices i and j,

$$M_{i,j} := \underline{M_{i,1}} \cap \overline{M_{1,j}}.$$

Applying $\breve{o}^{\mathcal{Q}}$ to each side of this equation we obtain

$$[o, M_{i,j}, o] := \overline{M_{i,1}} \cap \underline{M_{1,j}} = M_{j,i}$$

which shows the matrix is symmetric relative to $\breve{o}^{\mathcal{Q}}$.

5.20 **Proposition.** *Let $\mathcal{Q}$ be a quotient space of L. Then the map $L \ni x \to \overline{x}^{\mathcal{Q}}$ is a libra homomorphism with image $L_{\mathcal{Q}}$.*

Proof. For $x, y, z \in L$ and $B \in \mathcal{Q}$ we have by *Lemma* 5.15

$$[\overline{x}^{\mathcal{Q}}, \overline{y}^{\mathcal{Q}}, \overline{z}^{\mathcal{Q}}](\underline{B}) = \overline{x}^{\mathcal{Q}} \circ \overline{y}^{\mathcal{Q}^{-1}} \circ \overline{z}^{\mathcal{Q}}(\underline{B}) = \overline{x}^{\mathcal{Q}} \circ \overline{y}^{\mathcal{Q}^{-1}}(\overline{[z, B, B]}) =$$

$$\overline{x}^{\mathcal{Q}}(\underline{[[z, B, B], [z, B, B], y]}) = \overline{x}^{\mathcal{Q}}(\underline{[z, B, B, z, y]}) = \overline{[x, [z, B, B, z, y], [z, B, B, z, y]]} =$$

$$\overline{[x, y, z, B, B, z, y]} = \overline{[[[x, y, z], B, B], z, y]} = \overline{[[x, y, z], B, B]} = \overline{[x, y, z]}^{\mathcal{Q}}(\underline{B})$$

which proves that $L \ni x \to \overline{x}^{\mathcal{Q}}$ is a homomorphism and that $L_{\mathcal{Q}}$ is the set of equilibrium for a correspondence libra. That it is onto $L_{\mathcal{Q}}$ is trivial. *Q.E.D*

5.21 **Definitions.** Let $\mathcal{Q}$ be a quotient space of L. The correspondence libra having $L_{\mathcal{Q}}$ as its set of equilibrium is the **correspondence libra associated with** $\mathcal{Q}$.

We shall say that $\mathcal{Q}$ is a **separating** quotient space if, for all distinct x and y in L, there exists some $B \in \mathcal{Q}$ such that $x \in B$ and $y \notin B$.

5.22 **Corollary.** *Let $\mathcal{Q}$ be a distinctive and separating quotient space of L. Then the map $L \ni x \to \overline{x}^{\mathcal{Q}} \in L_{\mathcal{Q}}$ is an isomorphism of L onto $L_{\mathcal{Q}}$.*

In particular, every libra containing a distinctive and separating quotient space is isomorphic to a correspondence libra.

Proof. In view of *Proposition* 5.20, we need only show that the function is injective. Let $x, y \in L$ be distinct. Since $\mathcal{Q}$ is separating, there exists $B \in \mathcal{Q}$ such that $x \in B$ and $y \notin B$. We have

$$\overline{x}^{\mathcal{Q}}(\underline{B}) = \overline{[x, B, B]} = \overline{B}.$$

Since $\mathcal{Q}$ is distinctive, *Proposition* 2.33 implies that B is distinctive. The balanced set $[y, B, B]$ is a non-trivial left coset of B, so it can equal no right coset of B. This means that

$$\overline{B} \neq \overline{[y, B, B]} = \overline{y}^{\mathcal{Q}}(\underline{B})$$

which implies $\overline{x}^{\mathcal{Q}}(\underline{B}) \neq \overline{y}^{\mathcal{Q}}(\underline{B})$. *Q.E.D*

6 The Libra of Projectivities.

6.1 **Discussion.** In these final two sections we shall be concerned with correspondence libras of projective mappings from a 1-dimensional projective space P onto another 1-dimensional projective space Q, where the associated field $(\mathsf{F}, +, \cdot)$ is commutative and of characteristic not equal to 0. For general information about projective spaces, the reader is invited to consult [4] and [3].

There are several equivalent ways to define such spaces P and Q. Perhaps the most *direct definition* is to let P be $\mathsf{F} \cup \{\infty\}$, where ∞ is a point adjoined so as to have something equal to $\frac{\mathsf{p}}{0}$, $\mathsf{p} \neq 0$ — (here $\infty + \mathsf{p} = \infty$, $\infty - \mathsf{p} = \infty$, and $\infty \cdot \mathsf{p} = \infty$ for such p as well). In the context of this set P exists a *cross ratio*: for $\mathsf{w}, \mathsf{x}, \mathsf{y}, \mathsf{z} \in \mathsf{F}$ such that $\{\mathsf{w}, \mathsf{x}, \mathsf{y}, \mathsf{z}\}$ contains at least three distinct elements,

$$< \mathsf{w}, \mathsf{x}, \mathsf{y}, \mathsf{z} >:= \frac{\mathsf{w} - \mathsf{y}}{\mathsf{w} - \mathsf{z}} \cdot \frac{\mathsf{x} - \mathsf{z}}{\mathsf{x} - \mathsf{y}} \tag{15}$$

and it is preservation of this cross ratio which distinguishes *projective mappings* f from all others:

$$< f(\mathsf{w}), f(\mathsf{x}), f(\mathsf{y}), f(\mathsf{z}) > = < \mathsf{w}, \mathsf{x}, \mathsf{y}, \mathsf{z} > . \tag{16}$$

Another *conventional definition* is to let P be the set of rays passing through the origin of a 2-dimensional vector space over F. This has the advantage of avoiding the *ad hoc* point ∞ but the disadvantage of having no immediate parameterization by elements of F. An identification between the two definitions comes by introducing a (rectangular) coordinate system (x, y) on the vector space and associating to each ray the first coordinate of the point of the horizontal line $\mathsf{H} := \{(\mathrm{x}, 1) : \mathsf{x} \in \mathsf{F}\}$ through which that ray passes. Of course the ray parallel to H does not intersect H, and so this ray is associated to ∞. Linear automorphisms ϕ of this 2-space preserve rays through the origin, and so induce bijections of P. These mappings f induced by the linear automorphisms ϕ are known to be just the projective automorphisms. To compute f relative to our identification, we evaluate ϕ at a point of H and then find the intersection of H with with the ray through this value. If the coordinate representation of ϕ is given by

$$\phi((\mathsf{x}, \mathsf{y})) := (\mathrm{a} \cdot \mathrm{x} + \mathrm{b} \cdot \mathrm{y}, \mathrm{c} \cdot \mathrm{x} + \mathrm{d} \cdot \mathrm{y}) \quad (\forall \mathsf{x}, \mathsf{y} \in \mathsf{F}),$$

then the corresponding projective mapping f on $\mathsf{F} \cup \{\infty\}$ is given by

$$f(\mathrm{x}) = \frac{\mathrm{a} \cdot \mathrm{x} + \mathrm{b}}{\mathrm{c} \cdot \mathrm{x} + \mathrm{d}}. \tag{17}$$

For this reason, in the context of the *direct definition*, projective mappings are often referred to as *linear fractional transformations.* The linear fractional transformation given by *Display* 17 will be denoted $\boxed{\begin{matrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{matrix}}$. As is well-known, direct calculation shows that, for products of such,

$$\boxed{\begin{matrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{matrix}} \circ \boxed{\begin{matrix} \mathrm{r} & \mathrm{s} \\ \mathrm{t} & \mathrm{u} \end{matrix}} = \boxed{\begin{matrix} \mathrm{ar+bt} & \mathrm{as+bu} \\ \mathrm{cr+dt} & \mathrm{cs+du} \end{matrix}}. \tag{18}$$

.

In the case in which F is the field of complex numbers, linear fractional transformations are frequently known as Möbius transformations. We shall adopt Möb(P,Q) for the general case to denote the family of all projective mappings from P onto Q.

The family Möb(P,Q) of Möbius transformations is a correspondence set of equilibrium with domain P and codomain Q. In both the procedures for obtaining Möb(P,Q), just described, a rather complex structure was assigned to the domain and codomain (or to the context in which they were produced), and the projective mappings were defined to be just those mappings which preserved this structure.

We presently examine an alternate procedure, which *a priori* puts no structure on the domain and codomain.

6.2 **Definitions.** An **involution** is a self-inverse permutation of a set.

Let P be a set consisting of at least 4 points. Let I be a family of involutions of P satisfying the following conditions:

(i) $(\forall f, g \in I)$ $f \circ g \circ f$ *is in* I*;*

(ii) $(\forall \mathsf{w}, \mathsf{x}, \mathsf{y}, \mathsf{z} \in \mathsf{P} : \mathsf{w} \neq \mathsf{x} \neq \mathsf{y} \neq \mathsf{z} \neq \mathsf{w})(\exists! f \in I)$ $f(\mathsf{w}){=}\mathsf{y}$ *and* $f(\mathsf{x}){=}\mathsf{z}$*;*

(iii) $(\forall \mathsf{w} \in \mathsf{P})(\forall f, g, h \in I : f(\mathsf{w}) = g(\mathsf{w}) = h(\mathsf{w}))$ $f \circ g \circ h = h \circ g \circ f$.

Then P is a **meridian** and I its **defining family of involutions**.

6.3 **Notation.** Let P be a meridian and I its defining family of involutions. Let $\mathsf{x}, \mathsf{y} \in \mathsf{P}$ be generic We adopt the following notation:

$$\boldsymbol{I}_{\mathsf{y},\mathsf{x}} := \{x \in I : x(\mathsf{x}) = \mathsf{y}\};$$

$$(\boldsymbol{I} \circ \boldsymbol{I})_{\mathsf{y},\mathsf{x}} := \{x \in I \circ I : x(\mathsf{x}) = \mathsf{y}\}.$$

6.4 **Theorem.** *Let* P *be a meridian and* I *its defining family of involutions. Let* $\{0, 1, \infty\}$ *be three distinct points of* P *and define* F *to be the complement in* P *of the singleton* $\{\infty\}$*. Then we have*

(i) $(\forall \mathsf{w} \in \mathsf{F})(\exists! t_{(\mathsf{w})} \in I_{\infty,\infty} \circ I_{\infty,\infty})\ \ t_{(\mathsf{w})}(0) = \mathsf{w}$;

(ii) $(\forall \mathsf{w} \in \mathsf{F} : \mathsf{w} \neq 0)\ (\exists! d_{(\mathsf{w})} \in (I \circ I)_{0,\infty})\ \ d_{(\mathsf{w})}(1) = \mathsf{w}$.

For $\mathsf{x}, \mathsf{y} \in \mathsf{F}$ *we define*

(iii) $\mathsf{x} + \mathsf{y} := t_{(\mathsf{y})}(\mathsf{x})$.

For $\mathsf{x} \in \mathsf{F}$ *we define* $0 \cdot \mathsf{x} := 0$, $\mathsf{x} \cdot 0 := 0$, *and*

(iv) $(\forall \mathsf{x}, \mathsf{y} \in \mathsf{F} : \mathsf{x} \neq 0 \neq \mathsf{y})\ \mathsf{x} \cdot \mathsf{y} := d_{(\mathsf{x})}(\mathsf{y})$.

It follows that

(v) $(F, +, \cdot)$ *is a field of characteristic not equal to 0;*

(vi) *if* $\mathsf{P} = \mathsf{F} \cup \{\infty\}$ *is identified with the 1-dimenional projective space over* F *(*cf. Discussion *6.1* supra*), then* I *is the set of projective involutions on* P*;*

(vii) $I \circ I =$ *Möb(*P,P*).*

Conversely, if P *is the 1-dimensional projective space over a commutative field* F *of characteristic not equal to 0, and if* I *is the set of involutions in Möb(*P,P*), then* P *is a meridian with defining set of involutions* I*.*

Proof. Assertions (i), (ii),(v), and (vi), as well as the converse, follow from Theorems 4.42 and 4.43 of [1].

At the beginning of this section we noted that, if we view P as $F \cup \infty$, then a generic element of Möb(P,P) is of the form

$$f(\mathrm{x}) = \frac{\mathrm{a} \cdot \mathrm{x} + \mathrm{b}}{\mathrm{c} \cdot \mathrm{x} + \mathrm{d}}, (a \cdot d + b \cdot c \neq 0).$$

We shall construct elements g and h of I such that

$$g \circ h = f. \tag{19}$$

The condition for a linear fractional transformation $\boxed{\begin{matrix} \mathrm{r} & \mathrm{s} \\ \mathrm{t} & \mathrm{u} \end{matrix}}$ to be an involution is that its trace be zero: $\mathrm{r} + \mathrm{u} = 0$. In view of *Display* 18, we thus need to find two invertible matrices A and B of zero trace such that $AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We treat six cases.

Case I: $\mathrm{b} \cdot \mathrm{c} \neq 0$. $A := \begin{bmatrix} 0 & -1/\mathrm{c} \\ 1/\mathrm{b} & 0 \end{bmatrix}$, $B := \begin{bmatrix} \mathrm{b} \cdot \mathrm{c} & \mathrm{b} \cdot \mathrm{d} \\ -\mathrm{a} \cdot \mathrm{c} & -\mathrm{b} \cdot \mathrm{c} \end{bmatrix}$;

Case II: $\mathrm{b} = 0 \neq \mathrm{c},\ \mathrm{a} \neq \mathrm{d}$. $A := \begin{bmatrix} 1 & \frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}} \\ \frac{\mathrm{c} \cdot (\mathrm{d}-1)}{\mathrm{d}-\mathrm{a}} & -1 \end{bmatrix}$, $B := \begin{bmatrix} 1 & \frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}} \\ \frac{\mathrm{c} \cdot (\mathrm{a}-1)}{\mathrm{d}-\mathrm{a}} & -1 \end{bmatrix}$;

Case III: $\mathrm{b} = 0,\ \mathrm{a} = \mathrm{d}$. $A := \begin{bmatrix} \mathrm{a} & 0 \\ \mathrm{c} + \mathrm{a} & -\mathrm{a} \end{bmatrix}$, $B := \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$;

Case IV: $\mathrm{b} \neq 0 = \mathrm{c},\ \mathrm{a} \neq \mathrm{d}$. $A := \begin{bmatrix} 1 & \frac{\mathrm{b} \cdot (\mathrm{a}-1)}{\mathrm{d}-\mathrm{a}} \\ \frac{\mathrm{d}-\mathrm{a}}{\mathrm{b}} & -1 \end{bmatrix}$, $B := \begin{bmatrix} 1 & \frac{\mathrm{b}(\cdot \mathrm{d}-1)}{\mathrm{d}-\mathrm{a}} \\ \frac{\mathrm{d}-\mathrm{a}}{\mathrm{b}} & -1 \end{bmatrix}$;

Case V: $\mathrm{c} = 0,\ \mathrm{a} = \mathrm{d}$. $A := \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $B := \begin{bmatrix} \mathrm{a} & \mathrm{b} + \mathrm{a} \\ 0 & -\mathrm{a} \end{bmatrix}$;

Case VI: $\mathrm{c} = \mathrm{d} = 0$. $A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B := \begin{bmatrix} 0 & \mathrm{d} \\ \mathrm{a} & 0 \end{bmatrix}$. We have shown that (vii) holds. *Q.E.D*

6.5 **Remark.** From *Theorem* 6.4 we see that there is third way to define a 1-dimensional projective space P: as a meridian. This definition is more fundamental than the *direct* and *common definitions* described in *Discussion* 6.1, in that it is independent of any *a priori* algebraic operations. The purpose of the present section is to characterize which libras are isomorphic to correspondence libras with equilibrium sets Möb(P,Q).

6.6 **Definition.** Let P and Q be meridians, respectively, with defining families I_{P} and I_{Q} of involutions respectively. A bijection $\phi|\mathsf{P} \to \mathsf{Q}$ such that $\{\phi^{-1} \circ x \circ \phi : \ x \in I_{\mathsf{Q}}\} = I_{\mathsf{P}}$ is a **meridian isomorphism**. When such exists, P and Q are said to be **isomorphic meridians**.

6.7 **Definition.** The **polar A^∞** of a subset A of L is the set of all $x \notin A$ such that $[a, x, a] = x$ for all $a \in A$. The **polar $\mathbf{a}^\infty$ of a point a** is $\{a\}^\infty$.

6.8 **Example.** It follows immediately from the definition of an affine libra, that polars of subsets of affine libras are void.

6.9 **Motivation.** The Latin original for the word "pole" meant "stake", which was used to "fix" an object to the ground. The elements of a point polar o^∞ are just those elements "fixed" by the involution $L \ni x \to [o, x, o] \in L$.

6.10 **Definition.** A libra $(L, [,,])$ is a **Möbius libra** if there exists a distinctive separating quotient space $\mathcal{Q}$ and an element $o \in L$ for which the following three conditions are satisfied:
(i) *L is the minimal balanced subset of itself containing o^∞;*
(ii) $(\forall S \in \mathcal{Q})$ *$o^\infty \cap S$ is a balanced subset of L with at least two points;*
(iii) $(\forall S, T \in \mathcal{Q} : S$ is a mixed coset of $T, S \neq [o, T, o])$ *$S \cap T \cap o^\infty$ is a singleton.*
A distinctive separating quotient space $\mathcal{Q}$ of a Möbius libra L, for which there exists $o \in L$ such that (i), (ii), and (iii) hold, will be called **defining quotient space for L**.

6.11 **Observation.** *Definition* 6.10.iii begs the question: what happens when $S = [o, T, o]$? In this case, for $s = [o, t, o] \in S$, we have

$$t \in o^\infty \iff [o, t, o] = t \iff s = t$$

and so, by *Definition* 6.10.ii,
(iv) $(\forall S, T \in \mathcal{Q} : S$ is a mixed coset of $T, S = [o, T, o])$ *$S \cap T \cap o^\infty = T \cap o^\infty$ and is strictly larger than a singleton.*

6.12 **Symmetric Matrix Interpretation.** Let notation be as in *Definition* 6.10, and construct a symmetric matrix $\mathcal{M}$ of the elements of $\mathcal{Q}$ as in the third paragraph of 5.19 *supra.* Then *Proposition* 2.18 implies that distinct elements having either a common row or common column have void intersection.

Let $\mathcal{O}$ be the matrix of which each element consists of the intersection of o^∞ with the corresponding element of $\mathcal{M}$. Then *Definition* 6.10.ii states that the entries of $\mathcal{O}$ are all balanced sets with two or more points, *Definition* 6.10.iii states that the intersection of any two non- symmetric elements of $\mathcal{O}$ (whose rows and columns are distinct) is a singleton, and *Observation* 6.11.iv states that symmetric elements of $\mathcal{O}$ are identical.

6.13 **Theorem.** *Let $(L, [,,])$ be a Möbius libra with defining quotient $\mathcal{Q}$. Then*
(i) *$\mathsf{X}_{\mathcal{Q}}$ is a meridian with defining set of involutions $I := \{\chi \in \mathfrak{RM}(L_{\mathcal{Q}}) : \chi = \chi^{-1}\}$.;*
(ii) *$\mathsf{Y}_{\mathcal{Q}}$ is a meridian with defining set of involutions $J := \{\chi \in \mathfrak{LM}(L_{\mathcal{Q}}) : \chi = \chi^{-1}\}$.;*
(iii) *$\mathsf{X}_{\mathcal{Q}}$ and $\mathsf{Y}_{\mathcal{Q}}$ are isomorphic meridians;*
(iv) *L is isomorphic to the correspondence libra Möb($\mathsf{X}_{\mathcal{Q}}$,$\mathsf{Y}_{\mathcal{Q}}$).*
Conversely, if P *and* Q *are isomorphic meridians, then F:=Möb(*P,Q*) is the set of equilibrium for a correspondence Möbius libra with canonical quotient space $\mathcal{Q}(F)$, conditions (i-iii) of the* Definition *6.10 holding for* ***each*** *element o of F.*

Proof. Let o be as in *Definition* 6.10. Let S in $\mathcal{Q}$ be generic, and let $x, y \in L$ be outside S. Then $[x, S, y]$ is a mixed coset of S. From *Corollary* 5.22 follows that

$$L \ni x \to \overline{x}^{\mathcal{Q}} \in L_{\mathcal{Q}} \textit{ is an isomorphism of } L \textit{ onto } L_{\mathcal{Q}}. \tag{20}$$

Recalling *Proposition* 5.4, define

$$\gamma|L \ni x \to \gamma_x := \lambda(\mathfrak{l}_{o,x}) = \overline{o}^{\mathcal{Q}} \circ (\overline{x}^{\mathcal{Q}})^{-1} \in \mathfrak{LM}(L_{\mathcal{Q}}).$$

Let ι be the identity in $\mathfrak{LM}(L_Q)$. For $x \in L$, we have from *Proposition* 5.4 and *Lemma* 3.4

$$\gamma_x \circ \gamma_x = \lambda(\mathfrak{l}_{o,x}) \circ \lambda(\mathfrak{l}_{o,x}) = \lambda(\mathfrak{l}_{o,x} \circ \mathfrak{l}_{o,x}) = \lambda(\mathfrak{l}_{[o,x,o],x}).$$

It follows that

$$\gamma_x \circ \gamma_x = \iota \iff [o,x,o] = x \iff x \in o^\infty. \tag{21}$$

Let $\chi \in \mathfrak{LM}(L_Q)$ be generic. By *Proposition* 5.4 we know that χ is of the form $\lambda(\mathfrak{l}_{y,z})$ for some $y, z \in L$. From *Lemma* 3.4 we have

$$\chi = \lambda(\mathfrak{l}_{y,z}) = \lambda(\mathfrak{l}_{o,[o,y,z]}) = \gamma_{[o,y,z]}.$$

Thus γ is surjective and *Display* (21) implies

$$J = \{\chi \in \mathfrak{LM}(L_Q) : \chi = \chi^{-1}\} = \{\gamma(x) : x \in o^\infty\}. \tag{22}$$

.

Our immediate task is to demonstrate that J is a defining family of involutions for $\mathsf{P} := \mathsf{Y}_Q$. Since each element of J is self-inverse, *Definition* 6.2.i follows from

$$(\forall f, g \in J)(f \circ g \circ f)^{-1} = f^{-1} \circ g^{-1} \circ f^{-1} = f \circ g \circ f$$

and *Display* (22).

Let $\mathsf{x}, \mathsf{w}, \mathsf{y}, \mathsf{z} \in \mathsf{P}$ satisfy $\mathsf{x} \neq \mathsf{w} \neq \mathsf{y} \neq \mathsf{z} \neq \mathsf{x}$. For Z in $\overline{o}^{Q^{-1}}(\mathsf{z})$ and W in x, there exist $r, s \in L$ such that $[r, Z, s] = W$. For $z \in Z$ we have by *Lemma* 2.9

$$[r, z, Z] = [r, Z, z] = [r, Z, s, s, z] = [W, s, z] \in \overline{W} = \mathsf{x}$$

so $S := [r, z, Z]$ is in $\overline{o}^{Q^{-1}}(\mathsf{z}) \cap \mathsf{x}$. Similarly, there exists T in $\overline{o}^{Q^{-1}}(\mathsf{y}) \cap \mathsf{w}$. Now S is some coset $[u, T, v]$ of T. If u were in T, then S would be a right coset of T, so x and w being partitions of right cosets, would be identical. This is absurd, so u is not in T. Similarly, if v were in T, then $\overline{o}^{Q^{-1}}(\mathsf{z})$ and $\overline{o}^{Q^{-1}}(\mathsf{y})$ would be identical partitions of left cosets, which would imply that z=y: absurd. Hence v is not in T, which means that S is a mixed coset of T. From *Definition* 6.10.iii and *Observation* 6.11.iv we know that there exists an element k of $o^\infty \cap S \cap T$. Let $f := \gamma_k$. We have by *Lemma* 5.18

$$(\overline{k}^Q)^{-1}(\mathsf{x}) = (\overline{k}^Q)^{-1}(\overline{S}) = \underline{k}_Q(\overline{S}) = \underline{[S,S,k]} = \underline{S} = (\overline{o}^Q)^{-1}(\mathsf{z})$$

which implies that

$$f(\mathsf{x}) = \gamma_k(\mathsf{x}) = \overline{o}^Q \circ (\overline{k}^Q)^{-1}(\mathsf{x}) = \mathsf{z}.$$

Similarly, we have $f(\mathsf{w}) = \mathsf{y}$, which establishes the existence part of *Definition* 6.2.ii.

Suppose that $h \in J$ also satisfies $h(\mathsf{x}) = \mathsf{z}$ and $h(\mathsf{w}) = \mathsf{y}$. By *Display* 21 there exists $l \in o^\infty$ such that $\gamma_l = h$. Then

$$z = h(\mathsf{x} = \gamma_l(\mathsf{x}) = \overline{o}^Q \circ (\overline{l}^Q)^{-1}(\mathsf{x}) \implies (\overline{l}^Q)^{-1}(\mathsf{x}) = (\overline{o}^Q)^{-1}(\mathsf{z}),$$

which by *Lemma* 5.18 implies that $\underline{[S,S,l]} = \underline{S}$. It follows that, for some $m \in L$ and any $s \in S$,

$$S = [m, [S,S,l], [S,S,l]] = [m, l, S, S, S, S, l] = [[m, l, s], S, l].$$

Since S is distinctive, it follows from *Proposition* 2.31.vi that l is in S. Similarly, l is in T. Assume that $l \neq k$. Then *Definition* 6.10.iii and *Observation* 6.11.iv imply that $T = [o, S, o]$. From *Proposition* 5.16.i we have

$$\{[o, M, o] : M \in \overline{o}^{Q^{-1}}(\mathsf{z})\} = \overline{[o, S, S]}$$

and so, since $T = [o, S, o]$, it follows that $\overline{T} = \overline{[o, S, S]}$. But T is also in w, so

$$\mathsf{w} = \overline{[o, S, S]} = \{[o, M, o] : M \in \overline{o}^{\mathcal{Q}^{-1}}(\mathsf{z})\}. \tag{23}$$

On the other hand, for $D \in \mathsf{z}$,

$$\overline{o}^{\mathcal{Q}^{-1}}(\mathsf{z}) = \underline{[D, D, o]} = \{[m, o, D, D, o] : m \in L\}$$

which implies

$$\{[o, M, o] : M \in \overline{o}^{\mathcal{Q}^{-1}}(\mathsf{z})\} = \{[o, [m, o, D, D, o], o] : m \in L\} =$$

$$\{[D, D, [o, m, o]] : m \in L\} = \overline{D} = \mathsf{z}.$$

Consequently *Display* 23 implies that $\mathsf{w} = \mathsf{z}$, which is absurd. Hence $l = k$, whence $h = f$, which establishes the uniqueness part of *Definition* 6.2.ii.

Now let $\mathsf{w} \in \mathsf{P}$ and $f, g, h \in J$ be such that $f(\mathsf{w}) = g(\mathsf{w}) = h(\mathsf{w})$. Choose $x, y, z \in o^\infty$ such that $\gamma_x = f$, $\gamma_y = g$, and $\gamma_z = h$. Let $\mathsf{z} := f(\mathsf{w})$. We have

$$\mathsf{z} = \gamma_x(\mathsf{w}) = \overline{o}^{\mathcal{Q}} \circ (\overline{x}^{\mathcal{Q}})^{-1}(\mathsf{w}).$$

By *Proposition* 5.12.ii there exists a unique common element S of $(\overline{o}^{\mathcal{Q}})^{-1}(\mathsf{w}) \cap \mathsf{z}$. We have

$$\underline{S} = (\overline{o}^{\mathcal{Q}})^{-1}(\mathsf{z}) = (\overline{x}^{\mathcal{Q}})^{-1}(\mathsf{w}) = (\overline{x}^{\mathcal{Q}})^{-1}(\overline{S}) = \underline{[S, S, x]}$$

which implies that x is in S. Similarly y and z are in S. From *Display* 22 and *Definition* 6.10.ii follows that $[x, y, z]$ is in $S \cap o^\infty$. By *Display* (21) $\gamma_{[x,y,z]} = {\gamma_{[x,y,z]}}^{-1}$, which yields

$$h \circ g \circ f = h^{-1} \circ g^{-1} \circ f^{-1} = (f \circ h \circ g)^{-1} =$$

$${\gamma_{[x,y,z]}}^{-1} = \gamma_{[x,y,z]} = f \circ g \circ h.$$

This proves *Definition* 6.2.iii.

It is evident that the bijection $\overline{o}^{\mathcal{Q}}$ satisfies

$$\{(\overline{o}^{\mathcal{Q}})^{-1} \circ \chi \circ \overline{o}^{\mathcal{Q}} : \ \chi \in \mathfrak{LM}(L_{\mathcal{Q}})\} = \mathfrak{RM}(L_{\mathcal{Q}}),$$

which implies (ii) and

$$\overline{o}^{\mathcal{Q}} \text{ is a merdian isomorphism from } \mathsf{X}_{\mathcal{Q}} \text{ to } \mathsf{Y}_{\mathcal{Q}} \tag{24}$$

which yields (iii).

That (iv) holds now follows from *Display* (20).

Conversely, suppose that P and Q are isomorphic meridians. Let $o \in \text{Möb}(\mathsf{P}, \mathsf{Q})$ be generic. We must show that conditions (i-iii) of *Definition* 6.10 hold. Define

$$\pi | \text{Möb}(\mathsf{P}, \mathsf{P}) \ni x \to o^{-1} \circ x \in \text{Möb}(\mathsf{P}, \mathsf{P}).$$

This function is evidently an isomorphism of Möb(P,Q) onto Möb(P,P) sending o to the identity mapping i and o^∞ to the set $Inv(\mathsf{P}) := \{x \in \text{Möb}(\mathsf{P}, \mathsf{P}) : x \circ x = i\}$. It follows that to demonstrate that conditions (i-iii) of *Definition* 6.10 hold for o and Möb(P,Q), it will suffice to show that the conditions hold for i and Möb(P,P). Let $\mathcal{Q}$ denote the correspondence libra quotient space $\mathcal{Q}(\text{Möb}(\mathsf{P}, \mathsf{P}))$, a generic element of which is a set

$$(\text{Möb}(\mathsf{P}, \mathsf{P}))_{\mathsf{y},\mathsf{x}} := \{x \in \text{Möb}(\mathsf{P}, \mathsf{P}) : \ x(\mathsf{x}) = \mathsf{y}\}, \quad \mathsf{x}, \mathsf{y} \in \mathsf{P}.$$

We observe that, for $\mathsf{x}, \mathsf{y} \in \mathsf{P}$,

$$[i, (\text{Möb}(\mathsf{P}, \mathsf{P}))_{\mathsf{y},\mathsf{x}}, i] = (\text{Möb}(\mathsf{P}, \mathsf{P}))_{\mathsf{x},\mathsf{y}}.$$

In this context, conditions (i-iii) of *Definition* 6.10 become the following:

(25) $$\text{Möb}(\mathsf{P}, \mathsf{P}) \text{ is generated by } Inv(\mathsf{P});$$

(26) $$(\forall \mathsf{x}, \mathsf{y} \in \mathsf{P})(\forall a, b, c \in Inv(\mathsf{P}) : a(\mathsf{x}) = b(\mathsf{x}) = c(\mathsf{x}) = \mathsf{y}) \quad a \circ b \circ c \in I);$$

and

(27) $$(\forall \mathsf{x}, \mathsf{y}, \mathsf{a}, \mathsf{b} \in \mathsf{P} : \ \mathsf{x} \neq \mathsf{b} \neq \mathsf{y} \neq \mathsf{a} \neq \mathsf{x})(\exists! z \in \text{Möb}(\mathsf{P}, \mathsf{P})) \quad z(\mathsf{x}) = \mathsf{y} \ and \ z(\mathsf{a}) = \mathsf{b}.$$

We begin with *Display* (25). *Theorem* 6.4 implies that P may be identified with a one-dimensional projective space $\mathsf{F} \cup \infty$ where F is a field of characteristic different from 0. Relative to this identification we have

$$Inv(\mathsf{P}) = \{\boxed{\begin{matrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & -\mathsf{a} \end{matrix}} : \ \mathsf{a}, \mathsf{b}, \mathsf{c} \in \mathsf{F}\}.$$

Let $x \in \text{Möb}(\mathsf{P}, \mathsf{P})$ be generic. From *Theorem* 6.4.vii follows that $x = y \circ z$ for some $y, z \in Inv(\mathsf{P})$. By the above, this means that there exist $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{r}, \mathsf{s}, \mathsf{t} \in \mathsf{F}$ such that

$$x = \boxed{\begin{matrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & -\mathsf{a} \end{matrix}} \circ \boxed{\begin{matrix} \mathsf{r} & \mathsf{s} \\ \mathsf{t} & -\mathsf{r} \end{matrix}}.$$

Since this can be rewritten as

$$[\boxed{\begin{matrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & -\mathsf{a} \end{matrix}}, \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}}, \boxed{\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}}, \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}}, \boxed{\begin{matrix} \mathsf{r} & \mathsf{s} \\ \mathsf{t} & -\mathsf{r} \end{matrix}}]$$

where each of the entries is an element of $Inv(\mathsf{P})$, it follows that *Display* (25) holds.

Let $\mathsf{x}, \mathsf{y} \in \mathsf{P}$ be generic and let $a, b, c \in Inv(\mathsf{P})$ satisfy $a(\mathsf{x}) = b(\mathsf{x}) = c(\mathsf{x}) = \mathsf{y}$. From *Definition* 6.2.iii follows that $a \circ b \circ c = c \circ b \circ a$. Consequently

$$(a \circ b \circ c) \circ (a \circ b \circ c) = (a \circ b \circ c \circ c \circ b \circ a) = i$$

which establishes *Display* (26).

That *Display* (27) holds follows from *Definition* 6.2.ii. *Q.E.D*

7 Examples.

7.1 **Introduction.** In this concluding section of the paper we present some examples illustrating the mathematical concepts introduced *supra*. These examples are all within the context of 2- or 3-dimensional real projective space, and so particularly susceptible of the imagination.

7.2 **The Circle as Meridian.** Let P denote the projective plane (over the field of real numbers)—we shall refer to P as **the plane**. Topologically, there is only one type of conic in P (given two conics C and D in P, there is always a projective automorphism of P mapping C onto D). Of course, to view P we must first choose a line n to represent the the points "at infinity", and then a given conic will appear as a parabola if it is tangent to n, an ellipse if it does not intersect n, or a hyperbola if it intersects n at two points. To present a graphical illustration, we must not only select n, but also put a metric on the complement. Matters seem easiest to visualize if we choose n not intersecting the conic, and if we choose the metric

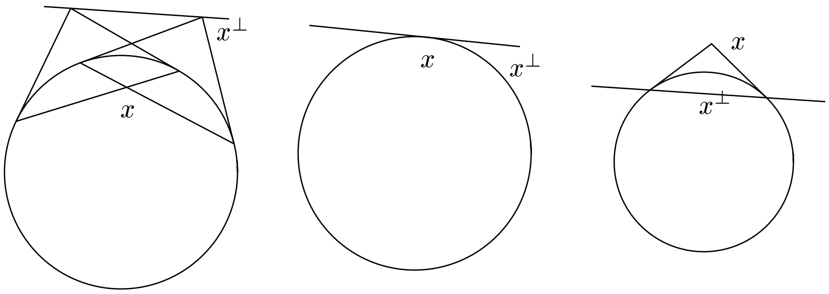

Figure 1: The line polar to a point (inside C, on C, and outside C).

such that the conic C is a circle. The complement of C, which we shall denote by I, has two connected components: one C^-, which contains no lines, is called the **inside** of C, and the other C^+, which does contain lines, is the **outside**.

The conic C induces a duality between points and lines. To each point x in the plane, a **polar** line $x^\perp$ is assigned as follows ((*cf.* [3] Chapter XIV.2 Poles and Polars)) :

(i) *Case I: $x \in C^+$. There are two lines tangent to C passing through x (which we call the **legs** of x) , and $x^\perp$ is the line passing through the two points of tangency;*

(ii) *Case II: $x \in C$. The tangent line is $x^\perp$;*

(iii) *Case III: $x \in C^-$. Each line through x intersects the conic at two points, and the lines tangent to these two points intersect at a point—the locus of all such points corresponding to lines through x is the polar line $x^\perp$.* (*cf.* Figure 1).

Points in I will be called **proper points**. A proper point x not only has a polar line, but also induces an involution $\hat{x}$ of the plane((*cf.* [3] Chapter XV.11 Involutions in the Plane)). By definition $\hat{x}(x) = x$ for any x. For x in the outside of C, and for other t in the plane, $\hat{x}(t)$ is as follows:

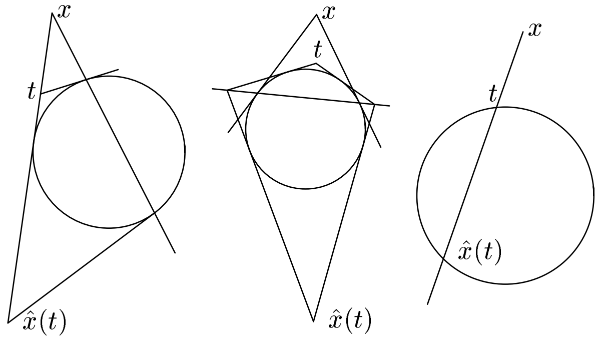

Figure 2: Involutions of the plane leaving C invariant: Cases I, II, and IV.

(iv) *Case I: $t \in C^+$ on a leg G of x. The point of tangency m of the leg of t distinct from G is on one line M through x—the line tangent to C, at the point in $M \cap C$ distinct from m, intersects G at the point $\hat{x}(t)$(*cf. *Figure 2);*

(v) *Case II: $t \in C^+$ not on a leg of x. The legs of t legs intersect $x^\perp$ at two points, and there is exactly one other point $\hat{x}(t)$ of which the legs intersect $x^\perp$ at the same two points(*cf. *Figure 2);*

(vi) *Case III: $t \in C$ on a leg G of x. Here $\hat{x}(t) = t$;*

(vii) *Case IV: $t \in C$ not on a leg of x. The point of C distinct from t on the line through t and x is $\hat{x}(t)$(*cf. *Figure 2);*

(viii) *Case V: $t \in C^-$. The points of $t^\perp$ are all outside the conic and so $\{\hat{x}(s) : s \in t^\perp\}$ is well-defined—this set is a line outside the conic and the point of which it is the dual is $\hat{x}(t)$.*

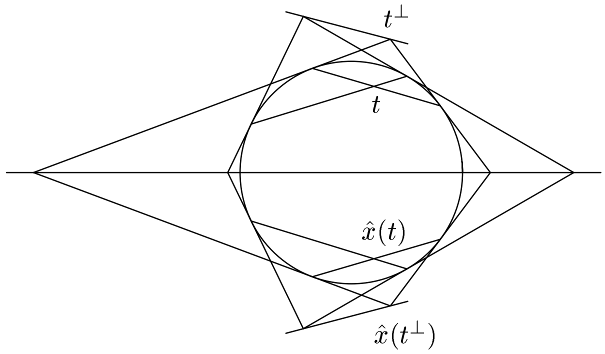

Figure 3: Involutions of the plane leaving C invariant: Case V.

(*cf.* Figure 3).

Suppose now that x is inside C. Then each point $y \in x^{\perp}$ is outside C and so using the above constructions we may find $\hat{y}(t)$. These points $\hat{y}(t)$ trace out a line as y runs through $x^{\perp}$—the pole of this line is $\hat{x}(t)$.

We shall show that C is a meridian with defining family $\{\hat{x}|_C : x \in I\}$ of involutions.

Proof. It is well-known that to each line g and point g not on g in the plane corresponds exactly one non-trivial line-preserving continuous involution of P having $\mathsf{g} \cup \{g\}$ for its set of fixed points. We note that, for x proper

(28) $\hat{x}$ is the unique projective involution such that $\{t : \hat{x}(t) = t\} = x^{\perp} \cup \{x\}$.

Furthermore, besides $x^{\perp}$, the only $\hat{x}$-invariant lines are those containing x.

We first verify *Definition* 6.2.i. We denote by K the set of all points of the plane left fixed by $\hat{x} \circ \hat{y} \circ \hat{x}$. Let x and y be proper points, and let t be a generic $\hat{y}$-fixed point. We have

$$\hat{x} \circ \hat{y} \circ \hat{x}(\hat{x}(t)) = \hat{x} \circ \hat{y}(t) = \hat{x}(t).$$

Conversely, if $\hat{x}(t)$ is fixed by $\hat{x} \circ \hat{y} \circ \hat{x}$, then

$$\hat{y}(t) = \hat{x} \circ \hat{x} \circ \hat{y} \circ \hat{x} \circ \hat{x}(t) = \hat{x}(\hat{x}(t)) = t.$$

It follows that

$$K = \{\hat{x}(t) : \hat{y}(t) = t\} = \{\hat{x}(y)\} \cup \hat{x}(y^{\perp}). \tag{29}$$

From the construction of polars described *supra* it is evident that bijective, line-preserving, mappings of the plane leaving C invariant also preserve polars. In particular,

$$\hat{x}(y^{\perp}) = (\hat{x}(y))^{\perp}.$$

Since $(\hat{x}(y))\hat{}$ has $\{\hat{x}(y)\} \cup (\hat{x}(y))^{\perp}$ as its set of fixed points, *Display* 29 implies that this set is just K. It follows from *Display* 28 that

$$\hat{x} \circ \hat{y} \circ \hat{x} = (\hat{x}(y))\hat{}$$

which establishes *Definition* 6.2.i.

Now let $w, x, y, z \in C$ satisfy the hypothesis of *Definition* 6.2.ii. Let m and n respectively, denote the lines determined by w and y, and by x and z, respectively (if $w = y$ or if $x = z$ we take m or n to be a line tangent to C). If u is the intersection of m with n, it follows from

Case IV supra that $\hat{u}(w) = y$ and $\hat{u}(x) = z$. This proves the *existence* part of *definition* 6.2.ii. Suppose that v is in I and that $\hat{v}$ also sends w to y and x to z. Then $\hat{v}$ leaves the lines m and n invariant. Consequently v is on both m and n, and so v must be u. It follows that *definition* 6.2.ii holds.

To prove *definition* 6.2.iii, we choose generic points $w, u \in C$, and elements $f, g, h \in I$ such that $u = \hat{f}(w) = \hat{g}(w) = \hat{h}(w)$, and show that

$$\hat{f} \circ \hat{g} \circ \hat{h} = \hat{h} \circ \hat{g} \circ \hat{f}.$$

We treat separately the cases $w \neq u$ and $w = u$.

Case I: $w \neq u$. We choose a line n outside C at random, but then choose a Cartesian coordinate system (x, y) on the vector space complement $V(\mathsf{n})$ of n in the plane such that C is the unit circle, that $w = (-1, 0)^{\rightharpoonup}$ and that $u = (1, 0)^{\rightharpoonup}$. Since $\hat{f}, \hat{g}$ and $\hat{h}$ all leave the X-axis invariant (but do not fix all the points on the X-axis), the points f, g, and h are all on the X-axis. If $a = (\mathrm{a}, 0)^{\rightharpoonup}$ is a generic point on the X-axis, direct computation shows that, for scalars s and r,

$$(\mathrm{x}, \mathrm{y}) \circ \hat{a}((\mathrm{r}, \mathrm{s})^{\rightharpoonup}) = \frac{(-(1 + \mathrm{a}^2)\mathrm{r} + 2\mathrm{a}, (\mathrm{a}^2 - 1)\mathrm{s})}{-2\mathrm{a}r + 1 + \mathrm{a}^2}.$$

Using this we compute, for another generic point $b = (\mathrm{b}, 0)^{\rightharpoonup}$ on the X-axis,

$$(\mathrm{x}, \mathrm{y}) \circ \hat{a} \circ \hat{b}((\mathrm{r}, \mathrm{s})^{\rightharpoonup}) =$$

$$\frac{(((1 + \mathrm{a}^2)(1 + \mathrm{b}^2) - 4\mathrm{ab})r - 2\mathrm{b}(1 + \mathrm{a}^2) + 2\mathrm{a}(1 + \mathrm{b}^2), (\mathrm{a}^2 - 1)(\mathrm{b}^2 - 1)s)}{(-2\mathrm{b}(1 + \mathrm{a}^2) + 2\mathrm{a}(1 + \mathrm{b}^2))r + (1 + \mathrm{a}^2)(1 + \mathrm{b}^2) - 4\mathrm{ab}} =$$

$$(\mathrm{x}, \mathrm{y}) \circ \widehat{(-b)} \circ \widehat{(-a)(r, s)}.$$

Applying this last result repeatedly, we obtain

$$(\hat{f} \circ \hat{g}) \circ \hat{h} = (-g)\hat{} \circ (-f)\hat{} \circ \hat{h} = (-g)\hat{} \circ (-h)\hat{} \circ (-(-f))\hat{}) = \hat{h} \circ \hat{g} \circ \hat{f}.$$

Case II: $w = u$. Here f, g, and h all lie on the line m tangent to C at w. We Let v be a point on m not on C, and let k be a point in the outside of C. Let x, y and z, respectively, be the lines joining f, g, and h respectively to k (*cf. Figure* 4). Let j_n be a sequence of

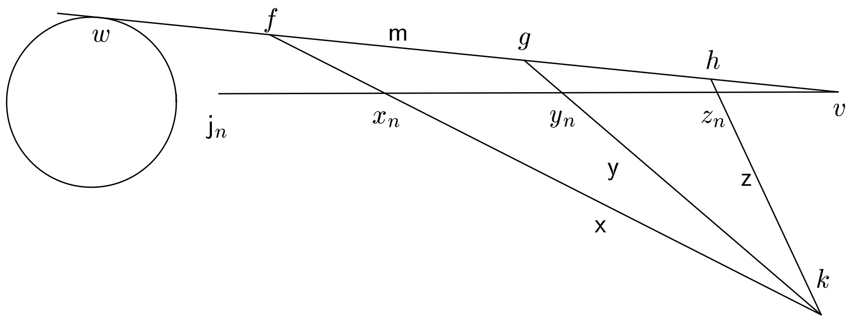

Figure 4: Illustration of Case II of the proof.

lines through v intersecting the line segment between k and h such that $\mathsf{j}_n \cap \mathsf{z}$ converges to h.

From Case I we know that, if x_n is the point of intersection of j_n with x, y_n is the point of intersection of j_n with y, and z_n is the point of intersection of j_n with z, then

$$\hat{x_n} \circ \hat{y_n} \circ \hat{z_n} = \hat{z_n} \circ \hat{y_n} \circ \hat{x_n}$$

for each n. Since x_n converges to f, y_n converges to g, and z_n converges to h, it follows that

$$\hat{f} \circ \hat{g} \circ \hat{h} = \hat{h} \circ \hat{g} \circ \hat{f}.$$

Consequently *Definition* 6.2.iii holds. It follows that C is a meridian. *Q.E.D*

Let (x, y) *be the coordinate system above, relative to which C is the unit circle. Letting* $\mathsf{0} := (1, 0)^{\rightarrow}$, $\mathsf{1} := (0, 1)^{\rightarrow}$, *and* $\infty := (-1, 0)^{\rightarrow}$, *we know from* Theorem *6.4 that the complement* F *of* $\{\infty\}$ *in C is a field. Ordering* F *in a counter-clockwise manner, it is not difficult to show that* F *is a well-ordered, order-complete, Archimedian ordered field, and thus isomorphic to the field of real numbers.*

7.3 **Spatial Context.** Let $\mathbf{S}$ be 3-dimensional real projective space. We shall refer to S simply as **space**. At times it is convenient to introduce coordinates to illustrate a situation or prove an assertion. Typically, depending upon the context, we shall select a hyperplane N (maximal proper projective subspace) for placement at "infinity". The complement of N is an affine space, and choosing a point o as "origin", the complement becomes a vector space $V(N)$. Then a Cartesian coordinate system $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ is chosen for $V(N)$: x, y, and z are linear functionals on $V(N)$ which simultaneously vanish at only the one point o.

Given a triple $(\mathrm{a}, \mathrm{b}, \mathrm{c})$ of real numbers, we shall denote by $(\mathrm{a}, \mathrm{b}, \mathrm{c})^{\rightarrow}$ the unique element v of $V(N)$ such that $(\mathrm{x}(v), \mathrm{y}(v), \mathrm{z}(v)) = (\mathrm{a}, \mathrm{b}, \mathrm{c})$. To specify points in all of space (including points in N), we sometimes employ *homogeneous coordinates* $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})^{\boldsymbol{homcoor}}$: $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})^{homcoor} := (\mathrm{a}/\mathrm{d}, \mathrm{b}/\mathrm{d}, \mathrm{c}/\mathrm{d})^{\rightarrow}$ for $\mathrm{d} \neq 0$ and $(\mathrm{a}/0, \mathrm{b}/0, \mathrm{c}/0)^{homcoor}$ is the intersection of N with the lines parallel to the ray from the origin o through $(\mathrm{a}, \mathrm{b}, \mathrm{c})^{\rightarrow}$.

Given a system $R(\mathrm{x}, \mathrm{y}, \mathrm{z})$ of equations or inequalities involving the coordinate functions, we shall write $\mathfrak{Sol}[\boldsymbol{R}(\mathrm{x}, \mathrm{y}, \mathrm{z})]$ for the set of points $(\mathrm{x}, \mathrm{y}, \mathrm{z})^{\rightarrow}$ such that $R(\mathrm{x}, \mathrm{y}, \mathrm{z})$ holds. For example, $\mathfrak{Sol}[\mathrm{x}^2 + \mathrm{y}^2 + \mathrm{z}^2 = 1]$ is the unit sphere in $V(N)$.

An invertible matrix $\begin{pmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} \\ \mathrm{e} & \mathrm{f} & \mathrm{g} & \mathrm{h} \\ \mathrm{i} & \mathrm{j} & \mathrm{k} & \mathrm{l} \\ \mathrm{m} & \mathrm{n} & \mathrm{o} & \mathrm{p} \end{pmatrix}$ defines a projective automorphism ϕ of space by letting the coordinates of $\phi((\mathrm{x}, \mathrm{y}, \mathrm{z})^{\rightarrow})$ be

$$\frac{(\mathrm{ax+by+cz+d}, \mathrm{ex+fy+gz+h}, \mathrm{ix+jy+kz+l})}{\mathrm{mx+ny+oz+p}}.$$

We shall adopt the notation $\boxed{\begin{matrix} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} \\ \mathrm{e} & \mathrm{f} & \mathrm{g} & \mathrm{h} \\ \mathrm{i} & \mathrm{j} & \mathrm{k} & \mathrm{l} \\ \mathrm{m} & \mathrm{n} & \mathrm{o} & \mathrm{p} \end{matrix}}$ for the automorphism ϕ. The reader will recall (or observe) that the composition of operators corresponds to the product of the corresponding matrices.

There are two essentially different types of quadric surfaces in space: one homeomorphic to a sphere and the other to a torus. We shall show how a sphere may be regarded as a meridian, and then how a torus serves as a nucleus of the libra of all projective transformations of the circle meridian.

Two objects in space are said to be **skew** if their intersection is void. In analogy with the previous example, we shall refer to points skew with the quadric surface as **proper points**. Points on the quadric surface will be referred to as **singular points**.

7.4 **The Sphere as Meridian.** Let $\boldsymbol{E}$ be a quadric surface in space homeomorphic to a sphere. The **outside** of E is the connected component $\boldsymbol{E}^{+}$ of the complement of the ellipsoid

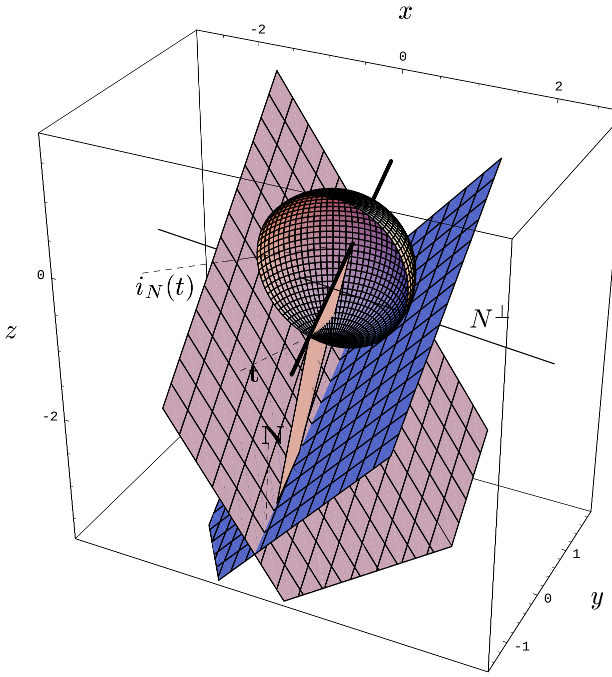

Figure 5: Dual Lines and their involution of the sphere.

which contains lines—the other component $\boldsymbol{E}^-$ is the **inside**. In our illustrations, the ellipsoid will appear as a sphere.

The conic E induces a duality between lines in space, as follows:

(i) *Case I:* g *is a line in* E^+. *There are two planes through* g *tangent to* E, *and the line through the points of tangency is* $\mathsf{g}^\perp$;

(ii) *Case II:* g *is tangent to* E. *Introducing a metric such that* E *is viewed as a sphere, the line through the point of tangency which is orthogonal to* g *is* $\mathsf{g}^\perp$;

(iii) *Case III:* g *intersects* E *at two points* a *and* b. *The line of intersection of the planes tangent to* E *at* a *and* b *is* $\mathsf{g}^\perp$. (*cf. Figure* 5).

Consider now a generic line n intersecting E in two points. The dual line $\mathsf{n}^\perp$ is entirely outside E. These two lines determine an involution ι_{n} of E as follows. The two points of $\mathsf{n} \cap E$ are fixed by ι_{n}: let x be a generic point on E skew from n. There exists exactly one plane A through n and x, and one plane B through $\mathsf{n}^\perp$ and x. Since n and $\mathsf{n}^\perp$ are skew, the two planes A and B are distinct. Hence $A \cap B$ is a line containing x, and so intersects E in exactly one other point—this is the point $\iota_{\mathsf{n}}(x)$(*cf. Figure* 5).

The ellipsoid E is a meridian with $\{\iota_{\mathsf{n}} : \mathsf{n}$ a line through the inside of E$\}$ its defining family of involutions. The proof is analogous to the proof of 7.2 and we omit it. We also note, without proof, that a field arising from the sphere meridian as in *Theorem* 6.4, is isomorphic with the field of complex numbers.

7.5 **The Elliptic Hyperboloid.** Let H denote a quadric surface not homeomorphic to a sphere—relative to any "plane at infinity" it appears as an elliptic hyperboloid (*cf.* Figure 6). We shall refer to H as **the hyperboloid**. An element x of S will be called a **proper point** if it is not on the hyperboloid, and a **singular point** if it is. We denote the complement of H in S, the set of all proper points, by $\boldsymbol{L}$. As a topological space, L has two components, which we shall denote $\boldsymbol{L^{ver}}$ and $\boldsymbol{L^{hor}}$. For a proper point x, we denote by $\boldsymbol{L_{(x)}}$ the component of L containing x.

Let $\boldsymbol{\Lambda}$ denote the family of all lines in space. A line l may be disjoint from H ($\mathsf{l} \in \boldsymbol{\Lambda_{skew}}$), tangent to H ($\mathsf{l} \in \boldsymbol{\Lambda_{tan}}$), or intersect H in two points ($\mathsf{l} \in \boldsymbol{\Lambda_{int}}$). Let $\mathcal{T}$ denote the usual topology on Λ: the family of all $\Sigma \subset \Lambda$ such that $\cup\{\mathsf{n} : \mathsf{n} \in \Sigma\}$ is open in S. Each $\mathsf{l} \in \Lambda_{skew}$ is in one component of L or the other. We write $\boldsymbol{\Lambda^{ver}_{skew}}$ for the set of l in L^{ver} and $\boldsymbol{\Lambda^{hor}_{skew}}$ for

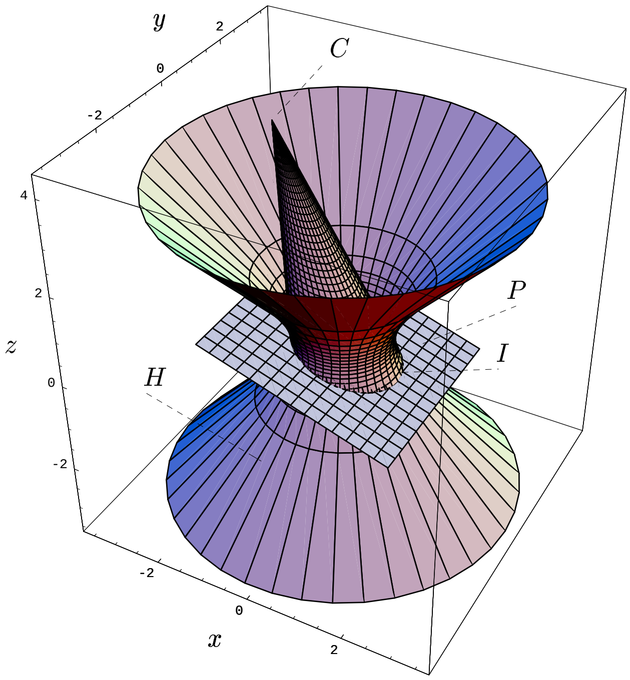

Figure 6: The elliptic hyperboloid $x^2 + y^2 = z^2 + 1$, a hyperboloid ellipse and cone, and a polar plane.

the set of l in L^{hor}. Then

(i) *Λ_{skew} is open in Λ and Λ_{skew}^{ver} and Λ_{skew}^{hor} are open and connected.*

Let $\boldsymbol{\Lambda_H}$ denote the lines (in $\boldsymbol{\Lambda_{tan}}$) which are subsets of H. These are called **rulings**:

(ii) *Λ_H is the disjoint union of two disjoint sets (called **reguli**), each homeomorphic to a circle(*cf. *7.8).*

We denote the two components of Λ_H by $\boldsymbol{P}$ and $\boldsymbol{Q}$:

(iii) *Λ_{tan} is connected compact and the complement in Λ_{tan} of Λ_H consists of two components, one of which is in the boundary of Λ_{skew}^{ver} (denoted $\boldsymbol{\Lambda_{tan}^{ver}}$), and the other in the boundary of Λ_{skew}^{hor} (denoted $\boldsymbol{\Lambda_{tan}^{hor}}$);*

(iv) *$\Lambda_H \cup \Lambda_{tan}^{ver}$ is the boundary of Λ_{skew}^{ver} and $\Lambda_H \cup \Lambda_{tan}^{hor}$ is the boundary of Λ_{skew}^{hor};*

(v) *Λ_{int} is connected open with boundary Λ_{tan}.*

Consider a proper point x. We write $\boldsymbol{\Lambda_x}$ for the family lines through x. The elements of $\Lambda_x \cap \Lambda_{tan}$ are the **legs of** x and their union constitutes a conical surface denoted $\boldsymbol{conicsurface(x)}$. We write $\boldsymbol{cone(x)}$ for the union of $conicsurfacex$ and $\cup\{\mathsf{l} \in \Lambda_x : \mathsf{l} \subset L_{(x)}\}$:

(vi) *$cone(x)$ is a compact cone with boundary $conicsurface(x)$;*

(vii) *$conicsurface(x) \cap H$ (denoted $\boldsymbol{conic(x)}$) is a conic section (which we call a **hyperbolic conic**).*

We write $\boldsymbol{x^\perp}$ for the plane determined by $conic(x)$ and denominate it the **polar** of x:

(viii) *each plane W not tangent to H is the polar of a unique proper point, denoted $\boldsymbol{W^\perp}$ and called the **pole** of W.*

The **polar of a singular point** t is the plane $\boldsymbol{t^\perp}$ tangent to H at t. As with proper points t is said to be the **pole** of its polar.

The relationship of pole to polar is a dual relationship ((*cf.* [3] XIV.9 Theorem 1)):

(ix) $(\forall x, y \in S)$ $x \in y^\perp \iff y \in x^\perp$.

Consider now a line $\mathsf{l} \in \Lambda$. We define the **polar of** l to be the intersection of all the planes $x^\perp$, $x \in \mathsf{l}$. This polar, denoted by $\mathsf{l}^\perp$ is also a line. Suppose now that ϕ is a projective automorphism of space. Clearly ϕ preserves the relation of tangency. Consequently

(x) *if H is ϕ-invariant, ϕ preserves polars, of points, lines, and planes.*
In particular,
(xi) *if both H and T are ϕ-invariant, where T is either a point, line, or plane in space, the $T^\perp$ is ϕ-invariant as well.*

7.6 **Proposition.** Let $C \subset L$ be a conic section. Then there exist exactly two proper points $a, b \in L$ such that $C \subset conicsurface(a)$ and $C \subset conicsurface(b)$. Furthermore, $C \subset L_{(a)} \cap L_{(b)}$.

Proof. Let N be a plane in space skew to C, and let the origin of $V(N)$ be in the plane determined by C, in the inside of C. Choose a coordinate system (x, y, z) such that $H = \mathfrak{Sol}[\mathrm{x}^2 + \mathrm{y}^2 = \mathrm{z}^2 + 1]$ and $C = \mathfrak{Sol}[\mathrm{x}^2 + \mathrm{y}^2 = 1/4, \mathrm{z} = 0]$. Let a be $(0, 0, \frac{1}{\sqrt{3}})^{\rightharpoondown}$ and b be $(0, 0, \frac{-1}{\sqrt{3}})^{\rightharpoondown}$. Direct calculation shows that $conic(a) = \mathfrak{Sol}[\mathrm{x}^2 + \mathrm{y}^2 = 4, \mathrm{z} = -\sqrt{3}]$, that $conic(b) = \mathfrak{Sol}[\mathrm{x}^2 + \mathrm{y}^2 = 4, \mathrm{z} = \sqrt{3}]$, and that $C \subset conicsurface(a) \cup conicsurface(b)$. Evidently a and b are in the same connected component of L: $\mathfrak{Sol}[\mathrm{x}^2 + \mathrm{y}^2 < \mathrm{z}^2 + 1]$.

Suppose that $C \subset conicsurface(c)$ for $c \in L$. Evidently c is not in N (for then the lines in $conicsurface(x)$, being all parallel and passing through C, would form a cylinder relative to the coordinate system (x, y, z), and no such cylinder could be tangent to H). Let l be the X-axis. The family of planes through l exhaust space and so x is in one of them M. Obviously x is not in the same plane as C (the place $z = 0$), so (x, z) is a coordinate system for M. The legs of c which pass through $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$ are tangent to $H \cap M$. The conic $H \cap M$ is symmetric relative to the Z-axis in M and, since $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$ are as well, c must lie on the Z-axis in M. Consequently, c must be in the YZ-plane of space S. In an analogous manner we can show that c must be in the XZ-plane of S. Thus c is on the Z-axis in S. Direct calculation now shows that c must be either a or b. *Q.E.D*

7.7 **Point Involutions Induced by the Hyperboloid.** Each non-coincident point x and plane X in space are the fixed points for a unique continuous involution of space (termed an *involutive homology with center x and central plane X*—(*cf.* [3] p.423)). Such an involution leaves the hyperboloid invariant, precisely when x is the pole of the plane X. In this case we shall denote the involution by $\hat{\boldsymbol{x}}$. We describe its action on a generic non-fixed point $s \in S$:
(i) *Case I: $s \in H$. If s is on a leg of x, $\hat{x}(s) = s$—otherwise the image $\hat{x}(s)$ is the point of H on the line through x and s distinct from s;*
(ii) *Case II: $s \notin H$. The conic surface $conicsurface(s)$ intersects the plane $x^\perp$ in a conic. By* Proposition *7.6 this conic is the intersection of $x^\perp$ with exactly one other conic surface $conicsurface(t)$—the image $\hat{x}(s)$ is just this element t.*
An image point $\hat{x}(s)$ is collinear with x and s and, in fact,

$$\hat{x}(\hat{x}(s)) = s. \tag{30}$$

We shall call the map $\hat{x}$ a **point involution**. We note that it follows from the definition of $\hat{x}$ that

$$(\forall N \text{ a plane in space}) \quad N \text{ is } \hat{x}\text{-invariant} \iff \text{either } N = x^\perp \text{ or } x \in N. \tag{31}$$

The composition of two distinct point involutions is not a point involution, nor is, in general, the product of three point involutions (*cf.* 7.10.ii *infra*). However we do have
(iii) $(\forall x, y \in L) \hat{x} \circ \hat{y} \circ \hat{x} = (\hat{x}(y))\hat{}$;
(iv) $(\forall \mathsf{l} \in \Lambda)(\forall x, y, z \in L \cap \mathsf{l})(\exists w \in L \cap \mathsf{l}) \quad \hat{w} = \hat{x} \circ \hat{y} \circ \hat{z}$.
Proof. Let x and y be proper points, and let s be an $\hat{y}$-fixed point. We have

$$\hat{x} \circ \hat{y} \circ \hat{x}(\hat{x}(s)) = \hat{x} \circ \hat{y}(s) = \hat{x}(s).$$

Conversely, if $\hat{x}(s)$ is fixed by $\hat{x} \circ \hat{y} \circ \hat{x}$, then

$$\hat{y}(s) = \hat{x} \circ \hat{x} \circ \hat{y}(s) \circ \hat{x} \circ \hat{x}(s) = \hat{x}(\hat{x}(s)) = s.$$

The function $\hat{x} \circ \hat{y} \circ \hat{x}$ evidently is a non-trivial involution of space, and so it is *the* such involution having $\{\hat{x}(s) : \hat{y}(s) = s\}$ for its set of fixed points. It follows that

$$\hat{x} \circ \hat{y} \circ \hat{x} = (\hat{x}(y))\hat{}$$

which establishes (iii).

Let $x, y, z \in L$ be on a single line l. Then l either is disjoint from H, is tangent to H, or intersects H in two points. In proving (iv) we treat these three cases separately.

Case I: $\mathsf{l} \in \Lambda_{skew}$. We choose a "plane at infinity" and a Cartesian coordinate system $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ for its complement such that l is the Z-axis and $H = \mathfrak{Sol}[\mathrm{x}^2 + \mathrm{y}^2 = \mathrm{z}^2 + 1]$. Direct calculation shows that

$$(32) \qquad (\forall \mathrm{z} \in \Re) \quad ((0, 0, \mathrm{z})^{\rightharpoondown})^{\perp} = \{(\mathrm{x}, \mathrm{y}, \text{-d}/\mathrm{z}, \mathrm{d})^{homcoor} : x, y, d \in \Re\}.$$

Recall that $\mathsf{l}^{\perp}$ is the intersection of all these planes:

$$\mathsf{l}^{\perp} := \{(\mathrm{x}, \mathrm{y}, 0, 0)^{homcoor} : x, y \in \Re\}.$$

Consider a projective automorphism of space:

$$\phi = \begin{bmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} \\ \mathrm{e} & \mathrm{f} & \mathrm{g} & \mathrm{h} \\ \mathrm{i} & \mathrm{j} & \mathrm{k} & \mathrm{l} \\ \mathrm{m} & \mathrm{n} & \mathrm{o} & \mathrm{p} \end{bmatrix}.$$

Then

$$(33) \qquad \mathsf{l}\ is\ \phi - invariant \iff \mathrm{cz{+}d{=}0{=}gz{+}h} \ \ (\forall \mathrm{z} \in \Re) \iff \mathrm{c} = \mathrm{d} = \mathrm{g} = \mathrm{h} = 0.$$

It follows from 7.5.xi and *Display* 32 that $\mathsf{l}^{\perp}$ is ϕ-invariant if both H and l are—consequently

$$(34) \qquad \mathrm{ix{+}jy{=}0{=}mx{+}ny} \ \ (\forall \mathrm{x{,}y} \in \Re) \implies \mathrm{i} = \mathrm{j} = \mathrm{m} = \mathrm{n} = 0.$$

If ϕ is a point involution for a point $z = (0, 0, \mathrm{z})^{\rightharpoondown}$ on l, then ϕ leaves each point of $z^{\perp}$ fixed. In this case we have

$$(35) \qquad (\forall x, y \in \Re) \ \ \mathrm{ax} + \mathrm{by} = \mathrm{x}, \ \mathrm{ex} + \mathrm{fy} = \mathrm{y} \implies \ \mathrm{a} = \mathrm{f} = 1, \ \mathrm{b} = \mathrm{e} = 0$$

. From *Displays* (33), 34, and 35 we have thatϕ is of the form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{k} & \mathrm{l} \\ 0 & 0 & \mathrm{o} & \mathrm{p} \end{bmatrix}$. ¿From the fact that ϕ is an involution follows that $p = -k$: $\phi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{k} & \mathrm{l} \\ 0 & 0 & \mathrm{o} & \text{-k} \end{bmatrix}$. Consider $x, y, z \in H$. We have $\phi((\mathrm{x}, \mathrm{y}, \mathrm{z})^{\rightharpoondown}) = (\mathrm{x}/(\mathrm{oz\text{-}k}), \mathrm{y}/(\mathrm{oz\text{-}k}), (\mathrm{kz{+}l})/(\mathrm{oz\text{-}k}))^{\rightharpoondown}$ so

$$\frac{\mathrm{x}^2}{(oz - k)^2} + \frac{\mathrm{y}^2}{(oz - k)^2} = 1 + \frac{\mathrm{kz{+}l}^2}{(oz - k)^2} \implies \mathrm{x}^2 + \mathrm{y}^2 = (\mathrm{kz{+}l})^2 + (oz - k)^2.$$

Since $1 + \mathrm{z}^2 = \mathrm{x}^2 + \mathrm{y}^2$, the above yields

$$1 + \mathrm{z}^2 = \mathrm{k}^2\mathrm{z}^2 + 2\mathrm{klz} + \mathrm{l}^2 + \mathrm{o}^2\mathrm{z}^2 - 2\mathrm{okz} + \mathrm{k}^2 \implies 1 = \mathrm{k}^2 + \mathrm{o}^2, \ \mathrm{kl{=}ko}.$$

Thus, in the case $\mathrm{k} \neq 0$, we have

$$(36) \qquad \phi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{k} & \mathrm{l} \\ 0 & 0 & \mathrm{l} & \text{-k} \end{bmatrix}, \ det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{k} & \mathrm{l} \\ 0 & 0 & \mathrm{l} & \text{-k} \end{pmatrix} = -\mathrm{k}^2 - \mathrm{l}^2 = -1.$$

We now check the fixed points $(x, y, z)^{\rightarrow}$ of ϕ:

$$x = \frac{x}{lz\text{-}k},\ y = \frac{y}{lz\text{-}k},\ z = \frac{kz+l}{lz\text{-}k}.$$

Through use of the quadratic equation on the third of these last equations, we obtain

$$kz+l = lz^2 - kz \quad \Longrightarrow \quad z = \frac{k \pm 1}{l}.$$

¿From the other two equations we see that, if x or y were non-zero, then lz-k=1. Hence

(37) the fixed points of ϕ are $(0, 0, \frac{k\text{-}1}{l})^{\rightarrow}$ and the points of the plane $\mathfrak{Sol}[z = \frac{k+1}{l}]$.

.

A composition of three transformations of the type in *Display* 36 returns another of the same type, which can be seen by direct matrix multiplication. It follows from *Display* 37 that this composition is a point involution, which proves (iv) for Case I.

Case II: $\mathsf{l} \in \Lambda_{tan}$. As in Case II of the proof of 7.2, (iv) for this case follows by taking a limit of lines of type Case I.

Case III: $\mathsf{l} \in \Lambda_{int}$. We choose a "plane at infinity" and a Cartesian coordinate system (x, y, z) for its complement such that l is the X-axis and $H = \mathfrak{Sol}[x^2 + y^2 = z^2 + 1]$. Direct calculation shows that, for $a \in \mathsf{l}$ such that $a = z(a)$,

$$\hat{a} = \boxed{\begin{array}{cccc} 1+a^2 & 0 & 0 & -2a \\ 0 & 1-a^2 & 0 & 0 \\ 0 & 0 & 1-a^2 & 0 \\ 2a & 0 & 0 & -1-a^2 \end{array}}.$$

It follows that the point involutions of points on the line l are the automorphisms of the form

$$\hat{a} = \boxed{\begin{array}{cccc} A & 0 & 0 & -B \\ 0 & C & 0 & 0 \\ 0 & 0 & C & 0 \\ B & 0 & 0 & -A \end{array}},\ A^2 = B^2 + C^2.$$

Let A,B,C,D,E,F,G,H,I $\in \Re$ satisfy $A^2 = B^2 + C^2$, $D^2 = E^2 + F^2$, and $H^2 = H^2 + I^2$: then

$$\boxed{\begin{array}{cccc} A & 0 & 0 & -B \\ 0 & C & 0 & 0 \\ 0 & 0 & C & 0 \\ B & 0 & 0 & -A \end{array}} \circ \boxed{\begin{array}{cccc} D & 0 & 0 & -E \\ 0 & F & 0 & 0 \\ 0 & 0 & F & 0 \\ E & 0 & 0 & -D \end{array}} \circ \boxed{\begin{array}{cccc} G & 0 & 0 & -H \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ H & 0 & 0 & -G \end{array}}$$

$$= \boxed{\begin{array}{cccc} K & 0 & 0 & -L \\ 0 & CFI & 0 & 0 \\ 0 & 0 & CFI & 0 \\ L & 0 & 0 & -K \end{array}}$$

for some L,K $\in \Re$. Noting that

$$A^2 = B^2 + C^2 \quad \Longleftrightarrow \quad C^2 + det\begin{pmatrix} A & 0 & 0 & -B \\ 0 & C & 0 & 0 \\ 0 & 0 & C & 0 \\ B & 0 & 0 & -A \end{pmatrix} = 0,$$

we see that to prove that (iv) holds for Case III, it will suffice to show that

$$(\mathrm{CFI})^2 + det \begin{pmatrix} \mathrm{K} & 0 & 0 & \text{-L} \\ 0 & \mathrm{CFI} & 0 & 0 \\ 0 & 0 & \mathrm{CFI} & 0 \\ \mathrm{L} & 0 & 0 & -\mathrm{K} \end{pmatrix} = 0.$$

But

$$det \begin{pmatrix} \mathrm{K} & 0 & 0 & \text{-L} \\ 0 & \mathrm{CFI} & 0 & 0 \\ 0 & 0 & \mathrm{CFI} & 0 \\ \mathrm{L} & 0 & 0 & -\mathrm{K} \end{pmatrix} =$$

$$det \begin{pmatrix} \mathrm{A} & 0 & 0 & \text{-B} \\ 0 & \mathrm{C} & 0 & 0 \\ 0 & 0 & \mathrm{C} & 0 \\ \mathrm{B} & 0 & 0 & -\mathrm{A} \end{pmatrix} \cdot det \begin{pmatrix} \mathrm{D} & 0 & 0 & \text{-E} \\ 0 & \mathrm{F} & 0 & 0 \\ 0 & 0 & \mathrm{F} & 0 \\ \mathrm{E} & 0 & 0 & -\mathrm{D} \end{pmatrix} \cdot det \begin{pmatrix} \mathrm{G} & 0 & 0 & \text{-H} \\ 0 & \mathrm{I} & 0 & 0 \\ 0 & 0 & \mathrm{I} & 0 \\ \mathrm{H} & 0 & 0 & -\mathrm{G} \end{pmatrix}$$

$$= (-\mathrm{C}^2)(-\mathrm{F}^2)(-\mathrm{I}^2) = -(\mathrm{CFI})^2$$

which produces the desired result. *Q.E.D*

7.8 **Hyperboloid Lines.** The quadric surface H is special, for it is the intersection of real space with a non-singular quadric surface in complex space which not only has "real trace", but "real rulings" ((*cf.* [3] p. 440). This means there are two families P and Q (called **reguli**), consisting of lines (called **rulings**) such that (*cf.* [3] Chapter XIV, Sections 8-11)

(i) *each ruling lies entirely in H and all lines in H are rulings;*

(ii) *no two rulings in the same regulus intersect;*

(iii) *two rulings $\mathsf{r} \in \mathsf{P}$ and $\mathsf{s} \in \mathsf{Q}$ intersect in exactly one point $\mathsf{t}(\mathbf{r}, \mathbf{s})$;*

(iv) *each singular point x lies on exactly two rulings $\mathbf{p}_x$ in P and $\mathbf{q}_x$ in Q;*

(v) *if E is a hyperbolic conic (*cf. *7.5.vii), each ruling of a regulus intersects E in one point and the mapping thus defined is a bijection of that regulus onto E;*

(vi) *each hyperbolic conic, inheriting a meridian structure from the plane it determines, induces a meridian structure on each regulus—on each regulus, the meridian structures thus induced are identical—hence each regulus may be regarded as a meridian;*

(vii) *a projective mapping of S onto itself which leaves H invariant, sends the elements of a regulus into the elements of a regulus—these induced mappings from one regulus to another are projective.*

The bijections in (v) being homeomorphisms, both P and Q are topologically indistinguishable from a circle. In view of (iii) and (iv) we have

(38) $\mathsf{P} \times \mathsf{Q} \ni (\mathsf{r}, \mathsf{s}) \to \mathsf{t}(\mathsf{r}, \mathsf{s}) \in H$ is a homeomorphism

which implies that H is topologically indistinguishable from a torus.

7.9 **Libra operation on L.** If ϕ is an involution of space equal to $\hat{x}$ for some $x \in L$, we shall write $\check{\phi}$ for x:

$$(\hat{x})^{\check{}} = x.$$

It follows from 7.7.iv that, for a line l not a member $\mathsf{P} \cup \mathsf{Q}$, the function

$$(\mathsf{l} \cap L) \times (\mathsf{l} \cap L) \times (\mathsf{l} \cap L) \ni (x, y, z) \to (\hat{x} \circ \hat{y}^{-1} \circ \hat{z})^{\check{}} \in \mathsf{l} \cap L$$

is a libra operation. The reader would find however, that if l were replaced by any subspace of L larger than a line, there would be elements x, y, z in that subspace such that $(\hat{x} \circ \hat{y}^{-1} \circ \hat{z})\hat{}$ failed to exist ((*cf.* 7.10.iii *infra*)). Nevertheless L **is** a libra if the operation is subtly changed. The change will be such that the new operation will agree with the old whenever the old exists (within L).

Each point involution $\hat{x}$ leaves H invariant. Hence, to find a meaning for $[x, y, z]$ for general $x, y, z \in L$, one is tempted seek some $w \in L$ such that $\hat{w}|_H = \hat{x}|_H \circ \hat{y}^{-1}|_H \circ \hat{z}|_H$. However, if such were found, then $\hat{x} \circ \hat{y}^{-1} \circ \hat{z} \circ \hat{w}^{-1}$ would be the identity on H—and any projective mapping equal to the identity on H is the identity on all of S. Thus it insufficient to look merely at restrictions to H of point involutions.

However, since a point involution $\hat{x}$ leaves H invariant and, being a projective mapping, preserves lines, it follows that the elements of $\mathsf{P} \cup \mathsf{Q}$ are preserved. Consider a line r in P. The image $\hat{x}(\mathsf{r})$ of r is the intersection of H with the plane determined by x and r. As noted in 7.8.i this image must be a line s in Q. The intersection $\mathfrak{t}(\mathsf{r}, \hat{x}(\mathsf{r}))$ being in both P and Q, its image by $\hat{x}$ must be as well:

$$\hat{x}(\mathfrak{t}(\mathsf{r}, \hat{x}(\mathsf{r}))) = \mathfrak{t}(\mathsf{r}, \hat{x}(\mathsf{r})). \tag{39}$$

We define

$$\tilde{x}|\mathsf{P} \ni \mathsf{r} \to \{\hat{x}(t) : t \in \mathsf{r}\} \in \mathsf{Q}. \tag{40}$$

The following equivalence is evident from the definitions: for $x \in L$, $\mathsf{r} \in \mathsf{P}$, and $\mathsf{s} \in \mathsf{Q}$

$$\tilde{x}(\mathsf{r}) = \mathsf{s} \iff x \in \mathfrak{t}(\mathsf{r}, \mathsf{s})^{\perp}. \tag{41}$$

In view of 7.5.ii this may be restated as

$$\tilde{x}(\mathsf{r}) = \mathsf{s} \iff \mathfrak{t}(\mathsf{r}, \mathsf{s}) \in x^{\perp}. \tag{42}$$

Since three points determine a plane $x^{\perp}$, it follows that the map $\tilde{x}$ is determined by its values at three distinct elements of P. Conversely, suppose that $r_1, r_2, r_3 \in \mathsf{P}$ and $s_1, s_2, s_3 \in \mathsf{Q}$ are pairwise distinct. Then $\mathfrak{t}(\mathsf{r}_1, \mathsf{s}_1), \mathfrak{t}(\mathsf{r}_2, \mathsf{s}_2), \mathfrak{t}(\mathsf{r}_3, \mathsf{s}_3)$ are non-collinear and so determine a plane W. The pole w of this plane leaves each of these three points fixed and so, by *Display* (41),

$$(\forall r_1, r_2, r_3 \in \mathsf{P}, s_1, s_2, s_3 \in \mathsf{Q} \textit{ pairwise distinct})$$

$$(\exists! w \in L) \;\; \tilde{w}(r_1) = s_1,\; \tilde{w}(r_2) = s_2,\; \tilde{w}(r_3) = s_3. \tag{43}$$

The foregoing suggests how to define a libra operation. Consider generic $x, y, z \in L$ and let $\mathsf{a}, \mathsf{b}, \mathsf{c} \in \mathsf{P}$ be distinct. Then the points $\tilde{x} \circ \tilde{y}^{-1} \circ \tilde{z}(a)$, $\tilde{x} \circ \tilde{y}^{-1} \circ \tilde{z}(b)$, and $\tilde{x} \circ \tilde{y}^{-1} \circ \tilde{z}(c)$ are distinct, which implies that $\mathfrak{t}(\mathsf{a}, \tilde{x} \circ \tilde{y}^{-1} \circ \tilde{z}(a))$, $\mathfrak{t}(\mathsf{b}, \tilde{x} \circ \tilde{y}^{-1} \circ \tilde{z}(b))$, and $\mathfrak{t}(\mathsf{c}, \tilde{x} \circ \tilde{y}^{-1} \circ \tilde{z}(c))$ are not collinear. By *Display* (43) there exists a unique $w \in L$ for which $\tilde{w}$ agrees with $\tilde{x} \circ \tilde{y}^{-1} \circ \tilde{z}$ on a,b, and c. This means that $\hat{x} \circ \hat{y}^{-1} \circ \hat{z} \circ \hat{w}^{-1}$ leaves each of the lines a, b, and c invariant. By 7.8.vi and the fact that a projective mapping on a meridian is determined by its values on three points, it follows that $\hat{x} \circ \hat{y}^{-1} \circ \hat{z} \circ \hat{w}^{-1}$ leaves each element of P invariant. This means that

$$(\forall x, y, z \in L)(\exists! w \in L) \;\; \tilde{w} = \tilde{x} \circ \tilde{y}^{-1} \circ \tilde{z}. \tag{44}$$

We define $\underline{[}$x,y,z$\underline{]}$ to be w:

$$\underline{[}x, y, z\underline{]}\tilde{} = \tilde{x} \circ \tilde{y}^{-1} \circ \tilde{z}. \tag{45}$$

For the remainder of this section, we shall view L as a libra with libra operation $\underline{[}, , \underline{]}$. From the foregoing it is evident that

$$\tilde{}|L \ni x \to \tilde{x} \in \text{Möb}(\mathsf{P}, \mathsf{Q}) \text{ is a libra isomorphism.}$$

7.10 **Balanced Subsets of L.** Let $\mathfrak{L}$ denote the family of all planes in space, and the $\mathfrak{L}_{tan}$ be the family of planes tangent to H. We have the following:

(i) $(\forall \mathsf{l} \in \Lambda) \quad \mathsf{l} \cap L$ *is balanced;*

(ii) *for each plane T in space,* $\quad T \in \mathfrak{L}_{tan} \Longleftrightarrow T \cap L$ *is balanced.*

Proof. Assertion (i) follows from 7.7.iv, since $\underline{[}x, y, z\underline{]}^{\sim}$ agrees with $x \circ y^{-1} \circ z$ whenever the latter exists as an element of L.

For $x, y, z \in T$, with T a plane tangent to H at a point $\mathsf{t}(\mathsf{p}, \mathsf{q})$, it follows from *Display* (41) that $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ all send p to q. Consequently $\underline{[}x, y, z\underline{]}^{\sim}$ sends p to q, and so 41 implies that $\underline{[}x, y, z\underline{]}^{\sim}$ is on T. It follows that $T \cap L$ is balanced.

Let T be a plane in space not tangent to H. We choose a Cartesian coordinate system for $V(T)$ such that the origin is $T^{\perp}$, $H = \mathfrak{Sol}[\mathrm{x}^2 + \mathrm{y}^2 = \mathrm{z}^2 + 1]$, and the lines in P rise in a counter-clockwise direction, where "counterclockwise" and "rise" are interpreted relative to *Figure* 7.5. Note the the planes dual to points in T are the planes containing the origin of $V(T)$. Let $x, y, z \in T$ be such that $x^{\perp}$ is the XZ-plane, $y^{\perp}$ is the YZ-plane, and $z^{\perp}$ is the plane determined by the points $(0,0,0)^{\rightharpoondown}$, $(1,0,1)^{\rightharpoondown}$, and $(0,1,1)^{\rightharpoondown}$.

$$(\forall \mathrm{x}, \mathrm{y}, \mathrm{z} \in \Re) \;\; \hat{x}((\mathrm{x},\mathrm{y},\mathrm{z})^{\rightharpoondown}) = (\mathrm{x},\text{-}\mathrm{y},\mathrm{z})^{\rightharpoondown} \textit{ and } \hat{y}((\mathrm{x},\mathrm{y},\mathrm{z})^{\rightharpoondown}) = (\text{-}\mathrm{x},\mathrm{y},\mathrm{z})^{\rightharpoondown}. \tag{46}$$

The line $\mathsf{p}_{(1,0,0)^{\rightharpoondown}}$ intersects $x^{\perp}$ at $(1,0,0)^{\rightharpoondown}$ so $\tilde{x}(\mathsf{p}_{(1,0,0)^{\rightharpoondown}}) = \mathsf{q}_{(1,0,0)^{\rightharpoondown}}$.We note that The line $\mathsf{q}_{(1,0,0)^{\rightharpoondown}}$ intersects $y^{\perp}$ in T and so $\tilde{y}^{-1}(\mathsf{q}_{(1,0,0)^{\rightharpoondown}}) = \mathsf{p}_{(-1,0,0)^{\rightharpoondown}}$. The line $\mathsf{p}_{(-1,0,0)^{\rightharpoondown}}$ intersects $z^{\perp}$ in T so $\tilde{z}(\mathsf{p}_{(-1,0,0)^{\rightharpoondown}}) = \mathsf{q}_{(1,0,0)^{\rightharpoondown}}$). Hence

$$\underline{[}z, y, x\underline{]}^{\sim}(\mathsf{p}_{(1,0,0)^{\rightharpoondown}}) = \mathsf{q}_{(1,0,0)^{\rightharpoondown}}.$$

Arguing similarly, we compute

$$\underline{[}z, y, x\underline{]}^{\sim}(\mathsf{p}_{(0,1,0)^{\rightharpoondown}}) = \mathsf{q}_{(0,1,0)^{\rightharpoondown}}.$$

Assume $w \in T$ is such that $\tilde{w} = \underline{[}z, y, x\underline{]}^{\sim}$. By the above, $w^{\perp}$ contains the origin and the points $(1,0,0)^{\rightharpoondown}$ and $(0,1,0)^{\rightharpoondown}$. Hence $w^{\perp}$ is the XY-plane. We thus have

$$\tilde{w}(\mathsf{p}_{(1/\sqrt{2},1/\sqrt{2})^{\rightharpoondown}}) = \mathsf{q}_{(1/\sqrt{2},1/\sqrt{2})^{\rightharpoondown}}. \tag{47}$$

On the other hand, *Display* 46 yields

$$\underline{[}z, y, x\underline{]}^{\sim}(\mathsf{p}_{(1/\sqrt{2},1/\sqrt{2},0)^{\rightharpoondown}}) = \tilde{z} \circ \tilde{y}^{-1} \circ \tilde{x}(\mathsf{p}_{(1/\sqrt{2},1/\sqrt{2},0)^{\rightharpoondown}}) = \tag{48}$$

$$\tilde{z} \circ \tilde{y}^{-1}(\mathsf{q}_{(1/\sqrt{2},-1/\sqrt{2},0)^{\rightharpoondown}}) = \tilde{z}(\mathsf{p}_{(-1/\sqrt{2},-1/\sqrt{2},0)^{\rightharpoondown}}).$$

Direct calculation shows that $\mathsf{p}_{(-1/\sqrt{2},-1/\sqrt{2},0)^{\rightharpoondown}}$ intersects $z^{\perp}$ at the point $(-2 - 1/\sqrt{2}, 2 - 1/\sqrt{2}, -2\sqrt{2})^{\rightharpoondown}$, which implies that the line $\tilde{z}(\mathsf{p}_{(-1/\sqrt{2},-1/\sqrt{2})^{\rightharpoondown}})$ does as well. But, by *Display* (47) and *Display* (48), this line is just $\mathsf{q}_{(1/\sqrt{2},1/\sqrt{2},0)^{\rightharpoondown}}$ and direct calculation shows that it intersects $z^{\perp}$ at the point $(2 + 1/\sqrt{2}, -2 + 1/\sqrt{2}, 2\sqrt{2})^{\rightharpoondown}$. This being absurd, w does not exist as earlier assumed. Thus T is not balanced, and (iii) holds. *Q.E.D*

7.11 **Planes Tangent to the Hyperboloid.** A singular point t, necessarily of the form $\mathsf{t}(\mathsf{p}, \mathsf{q})$ for $\mathsf{p} \in \mathsf{P}$ and $\mathsf{q} \in \mathsf{Q}$, has as its plane $\mathsf{t}(\mathsf{p}, \mathsf{q})^{\perp}$ of tangency the plane which intersects H precisely along the lines p and q. In particulular

(i) *each plane N in $\mathfrak{L}_{tan}$ intersects H along two lines, which lines intersect at the point of tangency $N^{\perp}$;*

If we fix the line p and let $t = \mathsf{t}(\mathsf{p}, \mathsf{q})$ move along p, the tangent planes $\mathsf{t}(\mathsf{p}, \mathsf{q})^{\perp}$ rotate completely around p and so exhaust the family of planes containing the line p. It follows that

(ii) *each plane containing a line in P is in $\mathfrak{L}_{tan}$;*

(iii) *the plane determined by a line $\mathsf{p} \in \mathsf{P}$ and a point x in L is in $\mathfrak{L}_{tan}$.*

For any x in a tangent plane $\mathsf{t}(\mathsf{p},\mathsf{q})^{\perp}$, it follows from 7.7.i that $\hat{x}$ leaves $\mathsf{t}(\mathsf{p},\mathsf{q})$ fixed, whence follows

(iv) $(\forall x \in \mathsf{t}(\mathsf{p},\mathsf{q})^{\perp})\ \ \tilde{x}(\mathsf{p}) = \mathsf{q}$.

Now let $x \in L$, $p \in \mathsf{P}$, and $q \in \mathsf{Q}$ be generic. The involution $\hat{x}$ maps $\mathsf{t}(\mathsf{p},\mathsf{q})^{\perp}$, the plane determined by p and q, onto the plane determined by $\hat{x}(\mathsf{p})$ and $\hat{x}(\mathsf{q})$:

(v) $\hat{x}(\mathsf{t}(\mathsf{p},\mathsf{q})^{\perp}) = \mathsf{t}(\hat{x}(\mathsf{q}),\hat{x}(\mathsf{p}))^{\perp}$.

In particular,

(vi) $\hat{x}(\mathsf{t}(\mathsf{p},\hat{x}(\mathsf{p}))^{\perp}) = \mathsf{t}(\mathsf{p},\hat{x}(\mathsf{p}))^{\perp}$.

It follows from *Display* (31) that

(vii) $x \in \mathsf{t}(\mathsf{p},\hat{x}(\mathsf{p}))^{\perp}$.

Let $\mathcal{Q}$ denote the set of all $N \cap L$ such that N is a plane tangent to H. Given $M = N \cap L$ in $\mathcal{Q}$, we shall denote N by $\overline{\boldsymbol{M}}$. From 7.10.ii we know that $\mathcal{Q}$ consists of balanced subsets of L. Let $M \in \mathcal{Q}$ and $x \in L$ be generic. For $m \in M$, the left coset $\lfloor x,m,M\rfloor$ is balanced and so 7.10.ii implies that $\overline{\lfloor x,m,M\rfloor}$ is in $\mathfrak{L}_{tan}$. By (i) there exist unique $p \in \mathsf{P}$ and $q \in \mathsf{Q}$ with $p \cup q \subset \overline{M}$. For generic $y \in M$, (iv) implies that $\hat{y}(\mathsf{p}) = \mathsf{q}$ and $\hat{m}(\mathsf{q}) = \mathsf{p}$—it follows that $\lfloor x,m,y\rfloor\tilde{(}\mathsf{p}) = x\tilde{(}\mathsf{p})$. From (vi) follows that each $\lfloor x,m,y\rfloor\tilde{(}\mathsf{p}) \in \mathsf{t}(\mathsf{p},\hat{x}(\mathsf{p}))^{\perp}$: that is

(viii) $\overline{\lfloor x,m,M\rfloor} = \mathsf{t}(\mathsf{p},\hat{x}(\mathsf{p}))^{\perp}$.

Thus

(ix) *the family of left cosets of an element of $\mathcal{Q}$ containing a line* p *in* P *is the family of elements $M \in \mathcal{Q}$ such that* p *is in* $\overline{M}$*: in other words, an abstract ordinate point is essentially the family of planes determined by a line* $\mathsf{p} \in \mathsf{P}$.

Reasoning analogously we obtain, for $\mathsf{q} \in \mathsf{Q}$,

(x) $\hat{x}(\mathsf{t}(\hat{x}(\mathsf{q}),\mathsf{q})^{\perp}) = \mathsf{t}(\hat{x}(\mathsf{q}),\mathsf{q})^{\perp}$;

(xi) $x \in \mathsf{t}(\hat{x}(\mathsf{q}),\mathsf{q})^{\perp}$;

(xii) *for* $\mathsf{q} \in \mathsf{Q}$ *with* $\mathsf{q} \subset \overline{M}$ $\quad \overline{\lfloor M,m,x\rfloor} = \mathsf{t}(\hat{x}(\mathsf{q}),\mathsf{q})^{\perp}$;

(xiii) *the family of right cosets of an element of $\mathcal{Q}$ containing a line* q *in* Q *is the family of elements $M \in \mathcal{Q}$ such that* q *is in* $\overline{M}$*: in other words, an abstract co-ordinate point is essentially the family of planes determined by a line* $\mathsf{q} \in \mathsf{Q}$.

We look at the three axioms for a Möbius libra with regard to L and $\mathcal{Q}$. Let o be any proper point. Consider $x \in L$. From 7.7.iii we know that $\hat{o} \circ \hat{x}^{-1} \circ \hat{o}$ exists and equals $\hat{o}(x)\hat{}$. This implies that $\lfloor o,x,o\rfloor$ equals $\hat{o}(x)$. Consequently we have

$$x \in o^{\infty} \iff \lfloor o,x,o\rfloor = x \iff \hat{o}(x) = x$$

which implies

$$o^{\infty} = o^{\perp}. \tag{49}$$

. From 7.10.ii we know that the first axiom holds:

L is the minimal balanced subset of itself containing o^{∞}.

¿From 7.7.iii-iv we know that the balanced subsets of L are the intersections of L with the lines in space, and the elements of $\mathcal{Q}$. Thus an element of $\mathcal{Q}$ intersects o^{∞} in a line, and so 7.7.iii yields the second axiom:

$(\forall S \in \mathcal{Q})$ $\ o^{\infty} \cap S$ is a non-trivial balanced subset of L.

Consider $S,T \in \mathcal{Q}$ such that S is a mixed coset of T. Note that 7.7.iii implies that $\lfloor o,T,o\rfloor = \hat{o}(T)$. Thus, if $S = \lfloor o,T,o\rfloor$, then the planes S and T intersect in a line in o^{∞}. Otherwise the line of intersections meets o^{∞} at a point. This extablishes the third axiom of a Möbius libra.

References

[1] *Kuruppu Ariyawansa, William Davidon, and Kelly McKennon*, A coordinate-free foundation for projective spaces treating projective maps from a subset of a vector space into another vector space, Technical Report, Department of Pure and Applied Mathematics, Washington State University, Pullman, WA 99164-3113 (submitted to Lecture Notes of the London Mathematical Society for publication).

[2] *Kelly McKennon*, A mathematical model of equilibrium, Laissez Faire City Times **4**, No. 46, (Nov. 13, 2000), http://www.zolatimes.com/writers/McKennon.html.

[3] *William Graustein*, Introduction to higher geometry. New York 1937.

[4] *Abraham Seidenberg*, Lectures in projective geometry. New York 1962.

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